Bimodule structure in the periodic $g\ell(1|1)$ spin chain

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Abstract

This paper is second in a series devoted to the study of periodic super-spin chains. In our first paper [1], we have studied the symmetry algebra of the periodic $g\ell(1|1)$ spin chain. In technical terms, this spin chain is built out of the alternating product of the $g\ell(1|1)$ fundamental representation and its dual. The local energy densities – the nearest neighbor Heisenberg-like couplings – provide a representation of the Jones–Temperley–Lieb (JTL) algebra JTL_N. The symmetry algebra is then the centralizer of JTL_N, and turns out to be smaller than for the open chain, since it is now only a subalgebra of $U_{\mathfrak{q}}s\ell(2)$ at $\mathfrak{q} = i$ – dubbed $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$ in [1]. A crucial step in our associative algebraic approach to bulk logarithmic conformal field theory (LCFT) is then the analysis of the spin chain as a bimodule over $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$ and JTL_N. While our ultimate goal is to use this bimodule to deduce properties of the LCFT in the continuum limit, its derivation is sufficiently involved to be the sole subject of this paper. We describe representation theory of the centralizer and then use it to find a decomposition of the periodic $g\ell(1|1)$ spin chain over JTL_N for any even N and ultimately a corresponding bimodule structure. Applications of our results to the analysis of the bulk LCFT will then be discussed in the third part of this series.

1 Introduction

The general philosophy of the lattice approach to LCFTs in the boundary case [2, 3] relies on the analysis of microscopic models – typically spin chains built out of alternating representations of a super Lie algebra such as $g\ell(m|n)$, with a nearest neighbour 'Heisenberg' coupling – as a bi-module over two algebras. In physical terms, one of these algebras is generated by the local hamiltonian densities, and the other is the 'symmetry' commuting with these hamiltonian densities.

It is natural to try to extend this approach [3] to the bulk case, but considerable mathematical difficulties are encountered in this endeavor. This is true even for the – a priori simplest – case of $g\ell(1|1)$, whose continuum limit is the ubiquitous symplectic fermion theory. The local hamiltonian densities then provide a (non-faithful) representation of the Jones–Temperley–Lieb algebra JTL_N. Its centralizer was studied in our previous paper [1], where we found that it is only a subalgebra – dubbed $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$ – of $U_{\mathfrak{q}}s\ell(2)$ at $\mathfrak{q} = i$ (recall that the centralizer of the ordinary Temperley–Lieb algebra in the open case is $U_i s\ell(2)$). The next step in the program consists thus in decomposing the spin chain as a bimodule over this $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$ and JTL_N – a rather technical task we tackle in this paper, leaving the discussion of the (many) physical implications to a sequel [4].

The plan of the paper is as follows. After preliminaries and reminders of various definitions in section 2, we explore the representation theory of the centralizer in section 3. The representation

theory of the Jones–Temperley–Lieb algebra is then summarized in section 4, largely based upon the seminal work of Graham and Lehrer [5]. Section 5 is devoted to the spin-chain decomposition over JTL_N. Considerable attention is paid to the absence of the double-centralizing property (a familiar aspect of the semi-simple case), and the ensuing technical complications for our analysis of the $g\ell(1|1)$ spin-chain. All these elements are put together in section 6 where the bimodule structure is finally obtained for the periodic model. The twisted model with antiperiodic boundary conditions is also decomposed as a (now, semisimple) bimodule over two centralizing algebras in the section 6. A few conclusios are gathered in section 7.

1.1   Notations

To help the reader navigate through this long paper, we provide a (partial) list of notations:

TL_N — the (ordinary) Temperley–Lieb algebra,

T_N^a — the periodic Temperley–Lieb algebra with the translation u, or the algebra of affine diagrams,

$JTL_N(m)$ — the Jones–Temperley–Lieb algebra with parameter m,

JTL_N — the Jones–Temperley–Lieb algebra at $m = 0$,

$\mathfrak{Z}_{JTL}$ — the centralizer of JTL_N in the $g\ell(1|1)$ spin chain,

$\pi_{g\ell}$ — the spin-chain representation of JTL_N,

$\rho_{g\ell}$ — the spin-chain representation of the quantum group $U_{\mathsf{q}}s\ell(2)$,

e, f — the renormalized powers,

$\mathsf{X}_{1,n}$ — the simple $U_i s\ell(2)$-modules,

$\mathsf{P}_{1,n}$ — the projective $U_i s\ell(2)$-modules,

X_n — the simple $U_i^{\mathrm{odd}} s\ell(2)$- and $\mathfrak{Z}_{JTL}$-modules

T_n — the indecomposable summands in spin-chain decomposition over the centralizer $\mathfrak{Z}_{JTL}$,

$\mathcal{W}_j$ — the standard modules over TL_N,

$\mathcal{P}_j$ — the projective modules over TL_N,

$\mathcal{L}_{j,(-1)^{j+1}}$ — the simple modules over JTL_N for which we also use the notation (d_j^0),

$\mathcal{W}_{j,(-1)^{j+1}}$ — the standard modules over JTL_N,

$\mathcal{W}_{j,e^{2iK}}$ — the standard modules over $JTL_N(m)$

$\overline{\mathcal{W}}_{0,\mathsf{q}^2}$ — the standard module over $JTL_N(m)$ for $j = 0$.

$\widehat{\mathcal{P}}_j$ — the indecomposable summands in spin-chain decomposition over JTL_N.

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2 Preliminaries

2.1 The Temperley–Lieb algebras in the periodic case

The models we are interested in have transfer matrices (hamiltonians) expressed in terms of Temperley–Lieb generators. In the periodic case, several variants of this algebra can be considered, and it is useful to start by going over a few definitions.

We begin with an algebra generated by the e_j's together with the identity, subject to the usual relations

$$
\begin{aligned}
e_j^2 &= me_j, \\
e_j e_{j\pm1} e_j &= e_j, \\
e_j e_k &= e_k e_j \qquad (j \neq k,\ k\pm1),
\end{aligned}
\tag{2.1}
$$

where $j = 1, \ldots, N$; m is a (real) parameter, and the indices are interpreted modulo N. This algebra is a quotient of the affine Hecke algebra of A-type and denoted by TL_N^a in the work of Graham and Lehrer [5, 6] whose definitions and notations we follow whenever possible. The algebra TL_N^a is also known as the periodic Temperley–Lieb algebra [7, 8].

The e_i's can be interpreted in terms of particular diagrams on an annulus (a representation which is known to be faithful [9]). A general basis element in the space of diagrams we will be interested in is obtained by taking N sites on the inner and N sites on the outer boundary of the annulus; these sites are connected in pairs, and only configurations that can be represented using lines inside the annulus without crossings are allowed. Diagrams related by an isotopy leaving the labeled sites fixed are considered equivalent. We call such (equivalence classes of) diagrams *affine* diagrams. Multiplication of two diagrams can be then defined by joining an inner to an outer annulus, and removing the interior sites. Whenever a closed contractible loop is produced in this multiplication, it is replaced by a numerical factor m. This defines abstractly an associative algebra which we denote as $T_N^a(m)$. Note that the diagrams in this algebra allow winding of through-lines around the annulus any integer number of times, and different windings result in independent algebra elements. Moreover, in the ideal of zero through-lines, any number of non-contractible loops is allowed. The algebra $T_N^a(m)$ is thus infinite-dimensional.

The action of the e_i generators in the diagram basis is well known [9]. Once in the sector with N through-lines of the diagram algebra T_N^a, we consider the generators u and u^{-1} of translations by one site to the right and to the left, respectively. The following additional defining relations are then obeyed,

$$
\begin{aligned}
u e_j u^{-1} &= e_{j+1}, \\
u^2 e_{N-1} &= e_1 \ldots e_{N-1},
\end{aligned}
\tag{2.2}
$$

and $u^{\pm N}$ is a central element. The algebra generated by the e_i and $u^{\pm1}$ with the defining relations (2.1) and (2.2) is isomorphic to $T_N^a(m)$ and called the *affine* Temperley–Lieb algebra.

We call *rank* [5] (see also [10]) of an affine diagram the minimal number of intersections with a radius of the annulus which does not contain any sites. The algebra TL_N^a introduced in (2.1) is spanned by all affine diagrams of even-rank in sectors with number of through-lines less than N and by the identity in the sector with N through-lines. Nothing is said at this stage about non contractible loops and windings of through-lines, and the algebra TL_N^a is also infinite dimensional.

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For the models we are interested in, with Hilbert spaces built out of (tensor products of) alternating representations, $N = 2L$ is even. Moreover, the pattern of representations forces one to consider translations by an even number of sites only, i.e. restrict to powers of u^2. This leads to a subalgebra $O_N(m) \subset T_N^a(m)$ spanned by all affine diagrams of even rank[1]. Physical applications require actually the consideration of further finite-dimensional quotients of the $O_N(m)$. The easiest way to define such quotients is to consider a homomorphism ψ to the Brauer algebra [11]. Recall first that the Brauer algebra is defined as the algebra of diagrams drawn inside a rectangle with lines connecting two identical or opposite edges, say the bottom and the top ones, with N sites on each of them and allowing any crossings, up to isotopy leaving the labeled sites fixed as usual. The image of $O_N(m)$ under the homomorphism ψ leads to diagrams that can be drawn in the annulus without crossings *plus* additional relations [12]: (i) non contractible loops are replaced by the same numerical factor m as for contractible loops; (ii) $u^N = 1$ (this allows one to 'unwind' through-lines of the affine diagrams); (iii) non-isotopic (in the annulus) diagrams connecting the same sites are identified. We call the resulting finite-dimensional algebra as the Jones–Temperley–Lieb algebra $JTL_N(m)$ (actually used in [2]), and it is the object we mostly want to study in this paper. This algebra was first introduced in [12] and called oriented annular subalgebra in the Brauer algebra.

For further references, we gather all the mentioned algebras in the diagram

$$TL_N \; \hookrightarrow \; TL_N^a \; \hookrightarrow \; T_N^a \; \longleftarrow \; O_N \; \xrightarrow{\;\psi\;} \; JTL_N \tag{2.3}$$

where we also introduced the notation for the open Temperley–Lieb algebra TL_N generated by e_j, for $1 \le j \le N - 1$; the arrows $\hookrightarrow$ denote embeddings of algebras while the doubled arrows denote projections (surjective homomorphisms of algebras).

We will only be concerned in this paper with the case $m = 0$ for which the algebra $JTL_{2L}(m)$ is non semi-simple; in the following we usually suppress all reference to m. We will also mostly restrict to a specific 'tensor product' representation known as a spin-chain model based on the $g\ell(1|1)$ algebra.

2.2   The closed $g\ell(1|1)$ super-spin chain

The closed $g\ell(1|1)$ super-spin chain [2, 1] is a tensor product representation $\mathcal{H}_N = \otimes_{j=1}^{N} \mathbb{C}^2$ of the algebra $JTL_N(0)$, which consists of $N = 2L$ tensorands labelled $j = 1, \dots, 2L$ with the fundamental representation of $g\ell(1|1)$ on even sites and its dual on odd sites. The representation of each e_j is given by projectors on the $g\ell(1|1)$-invariant in the product of two neighbour tensorands

$$e_j^{g\ell} = (f_j + f_{j+1})(f_j^\dagger + f_{j+1}^\dagger), \qquad 1 \le j \le 2L, \tag{2.4}$$

where we use a free fermion representation based on operators f_j and $f_j^\dagger$ acting non-trivially only on jth tensorand and obeying

$$\{f_j, f_{j'}\} = 0, \quad \{f_j^\dagger, f_{j'}^\dagger\} = 0, \quad \{f_j, f_{j'}^\dagger\} = (-1)^j \delta_{jj'}, \qquad f_{2L+1} = f_1, \quad f_{2L+1}^\dagger = f_1^\dagger, \tag{2.5}$$

where the minus sign for an odd j is due to presence of the dual representations of $g\ell(1|1)$.

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[1]The algebra $O_N(m)$ can be alternatively described as an algebra of diagrams with orientable lines (such that the arrows emanating from the even sites enter the odd sites on the inner boundary, and the reverse for the outer boundary), modulo odd-rank diagrams in the ideal without through-lines.

The generators $e_j^{g\ell}$ satisfy the (periodic) Temperley–Lieb algebra relations (2.1) with $m = 0$, and together with the generator u^2 translating the periodic spin-chain by two sites $j \to j+2$, it provides a representation of $JTL_{2L}(m = 0)$ which we denote by $\pi_{g\ell} : JTL_{2L}(0) \to \mathrm{End}_{\mathbb{C}}(\mathcal{H}_N)$. The representation $\pi_{g\ell}$ is known to be non-faithful [2].

The representation space $\mathcal{H}_{2L}$ is equipped with an inner product $\langle \cdot, \cdot \rangle$ such that $\langle f_j x, y \rangle = \langle x, f_j^\dagger y \rangle$ for any $x, y \in \mathcal{H}_{2L}$. We stress that the inner product is indefinite because of the sign factors in the relations (2.5). Then, the hamiltonian operator

$$H = -\sum_{j=1}^{2L} e_j^{g\ell}, \tag{2.6}$$

with the 'hamiltonian densities' $e_j^{g\ell}$ defined in (2.4), is self-adjoint $H = H^\dagger$ with respect to this inner product (actually, each $e_j^{g\ell}$ is a self-adjoint operator). Its eigenvalues are real and with eigenvectors are computed in [1] using a relation with XX spin-chains. As a consequence of the indefinite inner product, the self-adjoint hamiltonian has non-trivial Jordan cells (of rank-two).

2.3   Centralizers and bimodules

As was mentioned in the introduction, an important step in our approach is to find a decomposition of the spin-chain over the JTL_N for any finite N. The representation $\pi_{g\ell}$ of JTL_N is non-faithful and there are thus no direct evident ways in getting a decomposition of the spin-chain as it could be in the open case where it is a faithful representation of TL_N. For example, a general theory [13] of projective modules over a cellular algebra (which includes $TL_N(m)$ and $JTL_N(m)$ algebras) could be applied in a faithful representation. In our non-faithful case, we thus need an indirect way, which uses a symmetry algebra as it is explained below.

In general, an important concept in lattice models is the full symmetry algebra which is technically the centralizer of a 'hamiltonian densities' algebra of the model. By the latter algebra we generally mean any (representation of a) Hecke-type algebra – mostly $TL_N(m)$ for open spin-chains or $JTL_N(m)$ for closed ones. We recall that, for an associative algebra A and its representation space $\mathcal{H}$, *the centralizer* of A is an algebra $\mathfrak{Z}_A$ of the maximum dimension such that $[\mathfrak{Z}_A, A] = 0$, i.e., the centralizer is defined as $\mathfrak{Z}_A \cong \mathrm{End}_A(\mathcal{H})$ – the algebra of all endomorphisms on A-module $\mathcal{H}$.

Representation theory of the centralizer $\mathfrak{Z}_A$ is usually much easier to study than the representation theory of the 'hamiltonian densities' algebra A. It is thus more reasonable to start with a decomposition of spin-chains over $\mathfrak{Z}_A$ into indecomposable direct summands, which are technically tilting modules [14]. Then, studying all homomorphisms between the direct summands in the decomposition gives module structure over the 'hamiltonian densities' algebra A. In particular, multiplicities in front of tilting $\mathfrak{Z}_A$-modules give dimensions of simple A-modules, and subquotient structure of projective A-modules can be deduced from the one of the tilting $\mathfrak{Z}_A$-modules. As a result, we get a sequence of bi-modules $\mathcal{H}_N$ over the two commuting algebras parametrized by the number N of sites/tensorands in the spin-chain. We follow generally this strategy in the main part of the paper in studying representation theory of JTL_N and the corresponding bimodule.

As a simplest example, the open $g\ell(1|1)$ spin-chain exhibits a large symmetry algebra dubbed $\mathcal{A}_{1|1}$ in [2]. This algebra is the centralizer $\mathfrak{Z}_{\mathsf{TL}}$ of $TL_N(0)$ and is generated by the identity and the five
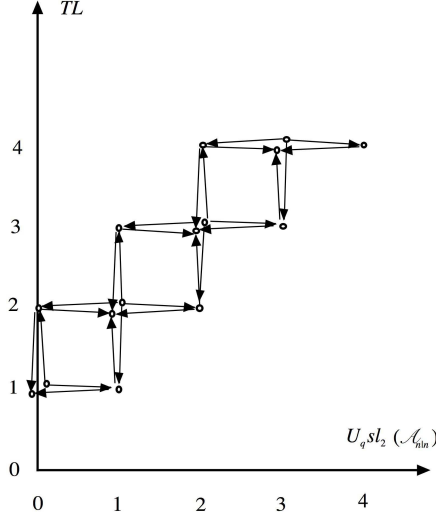
Figure 1: The structure of the open $g\ell(1|1)$ spin-chain for $N = 8$ sites, as a representation of $TL_N \boxtimes U_i s\ell(2)$. Some nodes with Cartesian coordinates $(n, n-1)$ occur twice and those nodes have been separated slightly for clarity.

generators

$$F_{(1)} = \sum_{1 \le j \le N} f_j, \qquad F^\dagger_{(1)} = \sum_{1 \le j \le N} f^\dagger_j,$$

$$F_{(2)} = \sum_{1 \le j < j' \le N} f_j f_{j'}, \qquad F^\dagger_{(2)} = \sum_{1 \le j < j' \le N} f^\dagger_{j'} f^\dagger_j, \qquad \mathrm{N} = \sum_{1 \le j \le N} (-1)^j f^\dagger_j f_j.$$

We note that these formulas give just a representation of the quantum group $U_{\mathfrak{q}} s\ell(2)$ for $\mathfrak{q} = i$. The fermionic generators, with the index '(1)', are from the nilpotent part and the bosonic ones form the $s\ell(2)$ subalgebra in $U_{\mathfrak{q}} s\ell(2)$ (see a precise correspondence below in (2.7) and (2.8).)

The decomposition of the open spin-chain as a bimodule over the pair $(TL_N, \mathcal{A}_{1|1})$ of mutual centralizers is shown on Fig. 1 for $N = 8$ case and borrowed from [3]. Each node with a Cartesian coordinate (n, n') in the bimodule diagram corresponds to a simple subquotient over the exterior product $TL_N \boxtimes U_i s\ell(2)$ and arrows show the action of both algebras – the Temperley–Lieb TL_N acts in the vertical direction (preserving the coordinate n), while $U_i s\ell(2)$ acts in the horizontal way. Indecomposable projective TL_N-modules $\mathcal{P}_{n'}$ (which are discussed below in Sec. 5.2) can be recovered by ignoring all the horizontal arrows, while tilting $U_i s\ell(2)$-modules $\mathsf{P}_{1,n}$ are obtained by ignoring all the vertical arrows of the bimodule diagram (these are also projective and given in (3.2).) Having the decomposition over $U_i s\ell(2)$, we see that the subquotient structure of direct summands over TL_N is obtained by drawing arrows corresponding to all possible homomorphisms between the tilting modules.

In the closed case, while the $g\ell(1|1)$ symmetry remains, being generated by $F_{(1)}$, $F^\dagger_{(1)}$, and N given above, the 'bosonic' $s\ell(2)$ generators $F_{(2)}$ and $F^\dagger_{(2)}$ do not commute with the action of $JTL_N(0)$. Instead, we have essentially only a 'fermionic' subalgebra of $\mathcal{A}_{1|1}$ that generates the centralizer of

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$JTL_N(0)$. We next describe in detail the centralizer of (the representation $\pi_{g\ell}$ of) JTL_N obtained first in our previous paper [1] where it is realized as a subalgebra of the quantum group $U_{\mathfrak{q}}s\ell(2)$.

2.4 The centralizer of $JTL_N(0)$

Recall first that the *full* quantum group $U_{\mathfrak{q}}s\ell(2)$ with $\mathfrak{q} = e^{i\pi/p}$ and an integer $p \geq 2$ is generated by E, F, $\mathsf{K}^{\pm 1}$, and e, f, h. The first three generators satisfy the standard quantum-group relations

$$\mathsf{KEK}^{-1} = \mathfrak{q}^2\mathsf{E}, \quad \mathsf{KFK}^{-1} = \mathfrak{q}^{-2}\mathsf{F}, \quad [\mathsf{E}, \mathsf{F}] = \frac{\mathsf{K} - \mathsf{K}^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}},$$

with additional relations

$$\mathsf{E}^p = \mathsf{F}^p = 0, \quad \mathsf{K}^{2p} = \mathbf{1},$$

and the divided powers $\mathsf{f} \sim \mathsf{F}^p/[p]!$ and $\mathsf{e} \sim \mathsf{E}^p/[p]!$ satisfy the usual $s\ell(2)$-relations:

$$[\mathsf{h}, \mathsf{e}] = \mathsf{e}, \qquad [\mathsf{h}, \mathsf{f}] = -\mathsf{f}, \qquad [\mathsf{e}, \mathsf{f}] = 2\mathsf{h}.$$

The full list of relations with comultiplication formulae are borrowed from [15] and listed in App. A where we also give relations to more usual (in the spin-chain literature) quantum group generators $S^{\pm}$, S^z and $\mathfrak{q}^{S^z}$. We will use in the text only the notation S^z which is proportional to h as $2\mathsf{h} = S^z$.

For applications to $g\ell(1|1)$ spin-chains, we consider only the case $p = 2$ and set in what follows $\mathfrak{q} \equiv i$. As a module over $U_{\mathfrak{q}}s\ell(2)$, the spin chain $\mathcal{H}_N$ is a tensor product of N copies of two-dimensional irreducibe representations defined as $\mathsf{E} \mapsto \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathsf{F} \mapsto \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\mathsf{K} \mapsto \mathfrak{q}\sigma^z = \begin{pmatrix} \mathfrak{q} & 0 \\ 0 & -\mathfrak{q} \end{pmatrix}$, and $\mathsf{e} \mapsto 0$, $\mathsf{f} \mapsto 0$. Using the $(N-1)$-folded comultiplications (A11), (A13), and (A14), we obtain the representation $\rho_{g\ell} : U_{\mathfrak{q}}s\ell(2) \to \mathrm{End}_{\mathbb{C}}(\mathcal{H}_N)$ in terms of the operators f_j and $f_j^{\dagger}$ defined in Sec. 2.2,

$$\rho_{g\ell}(\mathsf{h}) = \frac{1}{2}\sum_{j=1}^{N}(-1)^j f_j^{\dagger} f_j - \frac{L}{2}, \quad \rho_{g\ell}(\mathsf{e}) = \mathfrak{q}^{-1}\sum_{1 \leq j_1 < j_2 \leq N} f_{j_1}^{\dagger} f_{j_2}^{\dagger}, \quad \rho_{g\ell}(\mathsf{f}) = \mathfrak{q}\sum_{1 \leq j_1 < j_2 \leq N} f_{j_1} f_{j_2}, \quad (2.7)$$

$$\rho_{g\ell}(\mathsf{K}) = (-1)^{2\rho_{g\ell}(\mathsf{h})}, \qquad \rho_{g\ell}(\mathsf{E}) = \sum_{j=1}^{N} f_j^{\dagger} \rho_{g\ell}(\mathsf{K}), \qquad \rho_{g\ell}(\mathsf{F}) = \mathfrak{q}^{-1}\sum_{j=1}^{N} f_j. \qquad (2.8)$$

Definition 2.4.1. We now introduce an associative algebra $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$, with $\mathfrak{q} = i$. The algebra $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$ is generated by F_n, E_m $(n, m \in \mathbb{N} \cup \{0\})$, $\mathsf{K}^{\pm 1}$, and h with the following defining relations

$$\mathsf{KE}_m\mathsf{K}^{-1} = \mathfrak{q}^2\mathsf{E}_m, \qquad \mathsf{KF}_n\mathsf{K}^{-1} = \mathfrak{q}^{-2}\mathsf{F}_n, \qquad \mathsf{K}^4 = \mathbf{1}, \tag{2.9}$$

$$[\mathsf{E}_m, \mathsf{F}_n] = \sum_{r=1}^{\min(n,m)} P_r(\mathsf{h})\mathsf{F}_{n-r}\mathsf{E}_{m-r}, \tag{2.10}$$

$$\mathsf{E}_m\mathsf{E}_n = \mathsf{E}_n\mathsf{E}_m = 0, \quad \mathsf{F}_m\mathsf{F}_n = \mathsf{F}_n\mathsf{F}_m = 0, \quad [\mathsf{K}, \mathsf{h}] = 0, \tag{2.11}$$

$$[\mathsf{h}, \mathsf{E}_m] = (m + \tfrac{1}{2})\mathsf{E}_m, \qquad [\mathsf{h}, \mathsf{F}_n] = -(n + \tfrac{1}{2})\mathsf{F}_n, \tag{2.12}$$

where $P_r(\mathsf{h})$ are polynomials on h from the usual $s\ell(2)$ relation $[\mathsf{e}^m, \mathsf{f}^n] = \sum_{r=1}^{\min(n,m)} P_r(\mathsf{h})\mathsf{f}^{n-r}\mathsf{e}^{m-r}$, and we assume that $\sum_{r=1}^{0} f(r) = 0$.

The algebra $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$ has the PBW basis $\mathsf{E}_n\mathsf{F}_m\mathsf{h}^k\mathsf{K}^l$, with $n, m, k \geq 0$ and $0 \leq l \leq 3$. The positive Borel subalgebra is generated by h and E_n while the negative subalgebra – by h and F_n, for $n \geq 0$.

Remark 2.4.2. There is an injective homomorphism $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2) \to U_{\mathsf{q}}s\ell(2)$ of associative algebras defined as

$$\mathsf{E}_m \mapsto \mathsf{e}^m \mathsf{E} \frac{\mathsf{K}^2 + \mathbf{1}}{2}, \qquad \mathsf{F}_n \mapsto \mathsf{f}^n \mathsf{F} \frac{\mathsf{K}^2 + \mathbf{1}}{2}, \qquad m, n \geq 0. \tag{2.13}$$

This homomorphism together with expressions (2.7) and (2.8) define by restriction a representation of $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$ on the space $\mathcal{H}_N$ which we also denote by $\rho_{g\ell}$. The representation $\rho_{g\ell}$ of $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$ is given in [1] in terms of the fermionic operators f_j and $f_j^{\dagger}$.

We next recall the result [1] about the centralizer of the $JTL_{2L}(0)$.

Theorem 2.4.3. [1] Fix $\mathsf{q} = i$ and let $\mathfrak{Z}$ be the subalgebra in $\rho_{g\ell}\big(U_{\mathsf{q}}s\ell(2)\big)$ generated by $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$ and f^L, e^L. On the alternating periodic $g\ell(1|1)$ spin chain $\mathcal{H}_{2L}$, the centralizer $\mathfrak{Z}_{JTL}$ of the image of the Jones–Temperley–Lieb algebra $\pi_{g\ell}\big(JTL_{2L}(0)\big)$ is the associative algebra $\mathfrak{Z}$, where $\pi_{g\ell}$ is defined in (2.4).

We rely below on representation theory of the JTL_N-centralizer $\mathfrak{Z}_{JTL}$ in order to study the decomposition of the periodic spin-chain into indecomposable JTL_N-modules and what replaces the appealing bi-module structure known to exist in the open case when one turns to periodic systems is the subject of the following three sections.

3 Representation theory of the centralizer $\mathfrak{Z}_{JTL}$

We now briefly describe the representation theory of $\mathfrak{Z}_{JTL}$ (which coincides up to trivial features due to f^L, e^L with representation theory of $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$) . We begin with recalling the decomposition of the spin-chain $\mathcal{H}_{2L}$ over $U_{\mathsf{q}}s\ell(2)$ and then describe all simple subquotients over $\mathfrak{Z}_{JTL}$ occuring in the decomposition. We then use this in studying particular indecomposable modules constituting blocks in a spin-chain decomposition over the centralizer $\mathfrak{Z}_{JTL}$. We give a decomposition over $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$ in Sec. 3.3 and describe spaces of intertwining operators among indecomposable direct summands in the decomposition in Sec. 3.4, where we also give important facts about extensions ('glueings') among simple $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$-modules.

3.1 Spin-chain decomposition over $U_{\mathsf{q}}s\ell(2)$

We first recall a decomposition of $\mathcal{H}_{2L}$ over the full quantum group $U_{\mathsf{q}}s\ell(2)$, the open case [3], in the representation $\rho_{g\ell}$ defined in (2.7) and (2.8) (we suppress usually the notation $\rho_{g\ell}$ in the text below and write simply E instead of $\rho_{g\ell}(\mathsf{E})$, etc.)

$$\mathcal{H}_N\big|_{U_{\mathsf{q}}s\ell(2)} = \bigoplus_{j=1}^{L}(d_j^0) \boxtimes \mathsf{P}_{1,j}, \qquad N = 2L, \tag{3.1}$$

with multiplicities $d_j^0 = \sum_{i=j}^{L}(-1)^{j-i}\left(\binom{N}{L+i} - \binom{N}{L+i+1}\right)$ as dimensions of irreducibles over $TL_{2L}(0)$. The indecomposable direct summands $\mathsf{P}_{1,j}$ in the decomposition are projective covers of simple modules $\mathsf{X}_{1,j}$ which are introduced in App. B with the $U_{\mathsf{q}}s\ell(2)$-action given in (B1) (a module $\mathsf{X}_{1,j}$ has a trivial action of E, F, and K while it is a j-dimensional simple $s\ell(2)$-module.) We recall the subquotient

structure of $\mathsf{P}_{1,n}$ is then

$$
\mathsf{P}_{1,n} \quad = \quad
\begin{array}{ccc}
 & \mathsf{X}_{1,n} & \\
 & \swarrow \qquad \searrow & \\
\mathsf{X}_{1,n-1} & & \mathsf{X}_{1,n+1} \\
 & \searrow \qquad \swarrow & \\
 & \mathsf{X}_{1,n} &
\end{array}
\tag{3.2}
$$

with the $U_{\mathfrak{q}}s\ell(2)$-action explicitly described in App. B which is the particular case $\mathfrak{q} = i$ of [15]. In the diagram (3.2), we assume $\mathsf{X}_{1,0} \equiv 0$.

3.2 Simple modules over $U_{\mathfrak{q}}^{\mathbf{odd}}s\ell(2)$

We now describe simple modules over $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$ occurring in the spin-chain decomposition. Using Rem. 2.4.2, we consider the restriction to the subalgebra $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$ in a simple $U_{\mathfrak{q}}s\ell(2)$-module $\mathsf{X}_{1,r}$. The action (B1) on $\mathsf{X}_{1,r}$ where the generators E and F act trivially, and thus E_n and F_m do the same, proves the restriction decomposes onto one-dimensional subspaces

$$
\mathsf{X}_{1,r}|_{U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)} = \bigoplus_{n=1-r}^{r-1} \mathsf{X}_n,
$$

where we introduce a notation for simple modules X_n over $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$. The one-dimensional modules are parametrized by the weight n with respect to the Cartan generator $2\mathsf{h} = S^z$, where we also introduce the notation S^z for the Cartan generator familiar in spin-chain literature. We use two notations interchangeably in what follows.

With the use of the decomposition (3.1) and (3.2), we conclude that all the simple modules over $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$ that occur as subquotients in the spin-chain $\mathcal{H}_{2L}$ are the one-dimensional modules X_n parametrized by the weight n, where n is an integer number in the interval $-L \leq n \leq L$.

Simple modules over $\mathfrak{Z}_{JTL}$ are the same X_n for $-L+1 \leq n \leq L-1$ (e^L and f^L act trivially) and we use the same notation for them. The $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$-modules $\mathsf{X}_{\pm L}$ are combined by the action of e^L and f^L into a two-dimensional simple module over $\mathfrak{Z}_{JTL}$ which we also denote as X_L (avoiding a confusion we explicitly indicate the corresponding algebra in a decomposition). We thus conclude that the only difference in the representation theory of $\mathfrak{Z}_{JTL}$ is the two additional generators f^L and e^L map the two JTL_N-invariants on the opposite ends (at $S^z = \pm L$) of the spin-chain $\mathcal{H}_{2L}$ to each other. We next study modules over $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$ only, and a reader can easily recover the case of $\mathfrak{Z}_{JTL}$ with the use of the last comment.
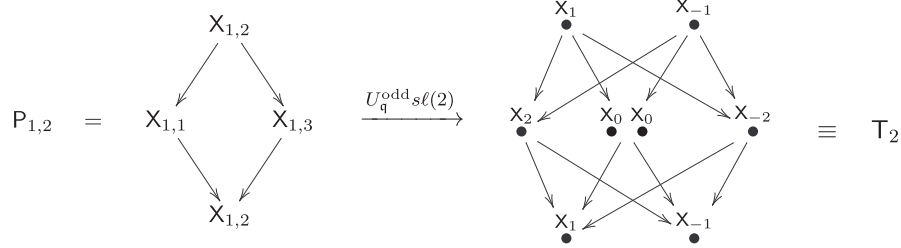
Remark 3.2.1. There are also simple $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$-modules of dimension $2r$ with the action given by the restriction on the $2r$-dimensional $U_{\mathfrak{q}}s\ell(2)$-modules $\mathsf{X}_{2,r}$, with $r \geq 1$, described in [15]. We do not give details because these modules do not appear in our spin-chains.

3.3 Spin-chain decomposition over $U_{\mathfrak{q}}^{\mathbf{odd}}s\ell(2)$

We now introduce indecomposable $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$-modules T_n which are used then in a decomposition of $\mathcal{H}_N$. With the use of the algebra homomorphism (2.13), we define modules T_n as the restriction of the projective $U_{\mathfrak{q}}s\ell(2)$-modules $\mathsf{P}_{1,n}$ described above in Sec. 3.1. It turns out that all T_n, with $1 \leq n \leq L$,

9

are indecomposable $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$-modules with dimension $4n$, where we use the homomorphism (2.13) together with the action in $\mathsf{P}_{1,n}$ from App. B.

As an example, for the restriction of the projective module $\mathsf{P}_{1,2}$ covering the doublet representation, we have the following diagrams of subquotient structure

$$
\mathsf{P}_{1,2} \;=\;
\begin{array}{c}
\mathsf{X}_{1,2} \\
\mathsf{X}_{1,1} \quad \mathsf{X}_{1,3} \\
\mathsf{X}_{1,2}
\end{array}
\quad\xrightarrow{\;U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)\;}\quad
\begin{array}{c}
\mathsf{X}_1 \qquad \mathsf{X}_{-1} \\
\mathsf{X}_2 \quad \mathsf{X}_0 \;\; \mathsf{X}_0 \quad \mathsf{X}_{-2} \\
\mathsf{X}_1 \quad \mathsf{X}_{-1}
\end{array}
\quad\equiv\quad \mathsf{T}_2
$$

where the horizontal arrow means the restriction to the subalgebra $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$ and the diagram on the right depicts the subquotient structure for T_2. The two-dimensional top subquotient $\mathsf{X}_{1,2}$ in $\mathsf{P}_{1,2}$ is splitted into two one-dimensional top subquotients $\mathsf{X}_{\pm 1}$ in T_2, and the arrows are splitted in a way that short south-*west* arrows[2], say mapping from X_1 to X_2, and south-*east* ones denote the action of $\mathsf{E} \equiv \mathsf{E}_0$ and $\mathsf{F} \equiv \mathsf{F}_0$, respectively, while the long south-west, say mapping from X_{-1} to X_2, and south-east arrows denote the action of E_1 and F_1, respectively. Due to (2.11) and the fermionic relations $\mathsf{E}_0^2 = \mathsf{F}_0^2 = 0$ in particular, we have that a node in the middle of the diagram, say the left X_0, have ingoing arrows of either south-west or south-east direction and its outgoing arrows are of opposite direction.

We next study a decomposition of the representation $\rho_{g\ell}$ of $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$ in $\mathcal{H}_N$ and begin with an example for $N = 8$ (or $L = 4$). The decomposition is given in Fig. 2, where the multiplicities d_n^0 are given by the same expression as the one after (3.1) for the open case because the restriction in each $\mathsf{P}_{1,n}$ is an indecomposable module over $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$ as we noted before. In this case, they are $(d_1^0) = (14)'$, $(d_2^0) = (14)$, $(d_3^0) = (6)$, and $(d_4^0) = (1)$ and must therefore be dimensions of simple modules over JTL_N. This will be confirmed below (these dimensions turn out to coincide with those of the simples in the open case, a peculiarity of this value of $\mathfrak{q} = i$). We note that a south-west arrow mapping from a subquotient X_m to X_n, i.e., $n > m$, represents an action of the raising generator $\mathsf{E}_{(n-m-1)/2}$ while a south-east arrow mapping from a subquotient X_m to X_n, i.e., $n < m$, represents an action of the lowering generator $\mathsf{F}_{(m-n-1)/2}$. We also note that all subquotients of T_n in the middle level (those X_k that satisfy $k - n = 0 \mod 2$) are divided into two classes – one having only south-west ingoing arrows and south-east outgoing ones, and in opposite, the second class has only south-east ingoing and south-west outgoing arrows. In Fig. 2, we thus have for T_4 that the left-most subquotient X_2 which is in the image of F_0 is mapped by E_0 to the X_3 in the bottom while all generators F_n represented by south-east arrows act as zero on it, and in opposite, the right-most node X_2 is sent to zero by E_0 while is mapped to three suquotients corresponding to the targets of the three south-east arrows.

In general, we have the decomposition over $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$ as

$$
\mathcal{H}_N\big|_{U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)} = \bigoplus_{n=1}^{L} (d_n^0) \boxtimes \mathsf{T}_n \tag{3.3}
$$

---

[2] We call any arrow *south-west* pointed down and left, and any arrow *south-east* that points down and right. We apologize to people from Southern Hemisphere where the south direction probably means 'up'.
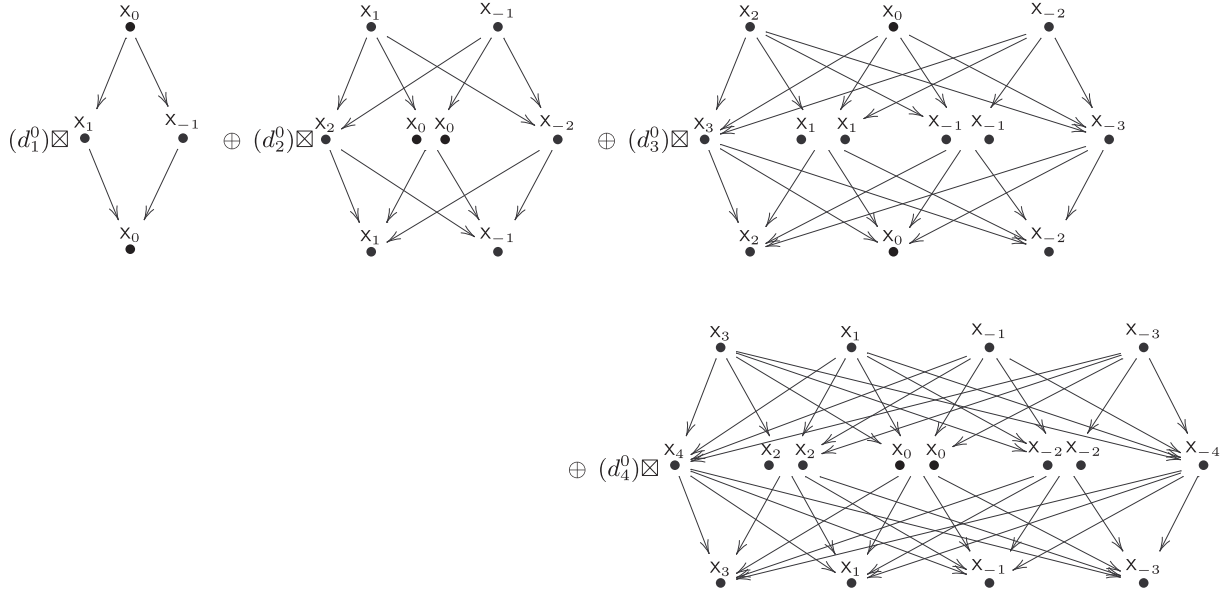
Figure 2: The decomposition of the spin-chain ($N = 8$) over $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$ into four indecomposable modules T_n with the multiplicities (d_n^0), $1 \le n \le 4$. Each node in the middle level of T_n have ingoing arrows only of one type (either south-west or south-east) and outgoing ones are of the opposite type.
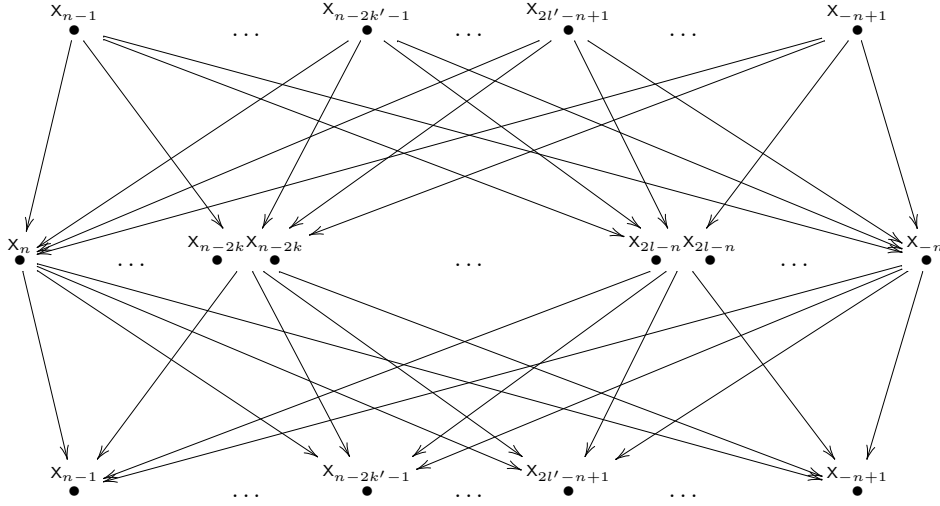

Figure 3: Subquotient structure of the $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$-modules T_n, where $n \ge 1$, $1 \le k, l \le n - 1$, $1 \le k', l' \le n - 2$. Each simple subquotient X_k appears once in the top and bottom parts of the diagram, and each X_k, with $-n < k < n$ and $k - n = 0 \bmod 2$, appears twice in the middle. A south-west arrow from X_m to X_n, i.e., when $n > m$, represents the generator $\mathsf{E}_{(n-m-1)/2}$ while a south-east arrow with $n < m$ corresponds to the action of $\mathsf{F}_{(m-n-1)/2}$.

with the subquotient structure for T_n, with $n \geq 2$, given in Fig. 3. We note that each node in the middle level of each T_n have ingoing arrows only of one type (either south-west or south-east) and outgoing ones are of the opposite type. This trivially follows from the relation (2.11) and the restriction on the subalgebra $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$ using formulas in App. B. With the use of the homomorphism (2.13), the formulas give an explicit action of E_n and F_m, with $n, m \geq 0$, in the basis used in App. B. We only note again that a south-west arrow mapping from a subquotient X_m to X_n, i.e., when $n > m$, represents an action of the raising generator $\mathsf{E}_{(n-m-1)/2}$ while a south-east arrow with $n < m$ corresponds to $\mathsf{F}_{(m-n-1)/2}$.

The space $\mathcal{H}_{2L}$ being considered as a module over the centralizer $\mathfrak{Z}_{JTL}$ has the same decomposition (3.3) with the only difference in the subquotient structure for $\mathsf{T}_{\pm L}$. The two nodes $\mathsf{X}_{\pm L}$ in Fig. 2 (for $L = 4$) and Fig. 3 are mixed by the action of f^L and e^L into one simple subquotient over $\mathfrak{Z}_{JTL}$.

Finally, we note that the full dimension of the $g\ell(1|1)$ spin chain is recovered via

$$\sum_{n=1}^{L} 4n \ d_n^0 = 4 \times 2^{2L-2} = 2^{2L} \tag{3.4}$$

in agreement with the dimension of the T_n being equal to $4n$. The same formula would represent the dimension of the $g\ell(1|1)$ Hilbert space in the open case, $4n$ now being the dimension of projective modules of the centralizer, equal to the full quantum group $U_{\mathfrak{q}} s\ell(2)$. These are replaced here by modules over $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$.

A reader can skip the rest of this section and next section in a first reading and go over directly to Sec. 5 where a decomposition over JTL_N is described.

3.4 Spaces of intertwining operators

Here, we describe all intertwining operators respecting the $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$ action on the spin-chain by studying homomorphisms among the indecomposable direct summands T_n in the decomposition (3.3) for each even N. We begin with basic information about first extension groups for a pair of simple modules. Then, we introduce Weyl-type modules that allow to describe images and kernels of all the homomorphisms between T_n.

3.4.1 Extensions for $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$

We study possible extensions between simple $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$-modules in order to construct indecomposable modules in what follows, and begin our description of the extensions with introducing some standard notations and definitions.

Let A and C be left $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$-modules. We say that a short exact sequence $0 \to A \to B \to C \to 0$ is an *extension* of C by A, and we let $\mathrm{Ext}^1_{U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)}(C, A)$ denote the set of equivalence classes (see, e.g., [16]) of extensions of C by A. Roughly speaking, by the extension group $\mathrm{Ext}^1(C, A)$ we simply mean a vector space of possible glueings between modules A and C into an indecomposable module B containing a submodule isomorphic to A and having at the top the subquotient C.

The following result easily follows from the relations (2.12) and the PBW basis given in Def. 2.4.1.

Proposition 3.4.2. *For $-L \leq n, m \leq L$, there are vector-space isomorphisms*

$$\mathrm{Ext}^1_{U_{\mathfrak{q}}^{odd} s\ell(2)}(\mathsf{X}_n, \mathsf{X}_m) \cong \begin{cases} \mathbb{C}, & n + m = 1 \mod 2, \\ 0, & otherwise. \end{cases}$$

All other first extensions between simple modules in the category of finite-dimensional $U_{\mathsf{q}}^{odd}s\ell(2)$-modules vanish.

3.4.3 Indecomposable and Weyl modules

Let N_{q}^+ denotes the positive subalgebra in $U_{\mathsf{q}}^{odd}s\ell(2)$ generated by E_n, for $n \geq 0$, and N_{q}^- denotes the negative subalgebra generated by F_n, with $n \geq 0$; let also B_{q}^+ denotes the positive Borel subalgebra generated by h and E_n, for $n \geq 0$, and B_{q}^- denotes the negative Borel subalgebra generated by h and F_n, with $n \geq 0$.

Using information about first-extension groups in Prop. 3.4.2, we construct two series of indecomposable $U_{\mathsf{q}}^{odd}s\ell(2)$-modules as extensions of two semi-simple modules. The first series consist of modules denoted by $\mathsf{B}_n^{(m)}$, where $n - m = 0 \mod 2$ and $n \geq 0$ and $m \geq 2$, with the trivial action of the positive subalgebra N_{q}^+ and having the subquotient structure

$$(3.5)$$

The second series consist of modules $\mathsf{R}_n^{(m)}$ with the trivial action of the negative subalgebra N_{q}^- and with the subquotient structure

$$(3.6)$$

Here, we use a notation for a representative from $\mathrm{Ext}^1_{U_{\mathsf{q}}^{odd}s\ell(2)}(\mathsf{X}_l, \mathsf{X}_{l'})$ depicted by a south-west arrow if $l' > l$ and a south-east arrow if $l' < l$ (dash lines are used just for clarity). We note that the source and the target of an arrow uniquely define a generator represented by the arrow. The generators E_k of the positive subalgebra are represented in (3.6) by south-west arrows mapping from a node X_l to $\mathsf{X}_{l'}$ whenever $(l' - l - 1)/2 = k$, and the action of F_k from the negative subalgebra is given in (3.5) by south-east arrows mapping from a node X_l to $\mathsf{X}_{l'}$ whenever $(l' - l + 1)/2 = -k$.

The modules $(\mathsf{B}_n^{(n+2)})$ $\mathsf{B}_n^{(n)}$ and $(\mathsf{R}_n^{(n+2)})$ $\mathsf{R}_n^{(n)}$ play a role of the (contragredient) Weyl modules over $U_{\mathsf{q}}s\ell(2)$. In more detail, the modules $\mathsf{B}_n^{(n)} \oplus \mathsf{X}_{-n}$ and $\mathsf{R}_n^{(n)} \oplus \mathsf{X}_n$ are restrictions of the Weyl module of dimension $2n+1$ to the negative and positive Borel subalgebras B_{q}^- and B_{q}^+ of $U_{\mathsf{q}}^{odd}s\ell(2)$, respectively. It is straightforward to check with the use of the defining relations (2.9)-(2.12) that these restrictions

are $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$-modules as well. Similarly, $\mathsf{B}_n^{(n+2)} \oplus \mathsf{X}_{n+1}$ and $\mathsf{R}_n^{(n+2)} \oplus \mathsf{X}_{-n-1}$ are restrictions of the contragredient Weyl $U_{\mathfrak{q}}s\ell(2)$-module of dimension $2n+3$. We show below that these 'Weyl' $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$-modules are building blocks of the spin-chain – indecomposable direct summands are glueings of a pair of these modules – as the Weyl modules over $U_{\mathfrak{q}}s\ell(2)$ do in the open case.

Using (3.5) and (3.6), we give the following filtrations of the $\mathsf{B}_n^{(m)}$ and $\mathsf{R}_n^{(m)}$ modules.

$$0 = \mathsf{B}_{-m+4}^{(m)} \subset \mathsf{B}_{-m+2}^{(m)} \subset \cdots \subset \mathsf{B}_{n-4}^{(m)} \subset \mathsf{B}_{n-2}^{(m)} \subset \mathsf{B}_n^{(m)}, \tag{3.7}$$

$$0 = \mathsf{R}_{-m+4}^{(m)} \subset \mathsf{R}_{-m+2}^{(m)} \subset \cdots \subset \mathsf{R}_{n-4}^{(m)} \subset \mathsf{R}_{n-2}^{(m)} \subset \mathsf{R}_n^{(m)}, \tag{3.8}$$

where for each pair of neighbour terms $\mathsf{B}_k^{(m)}/\mathsf{B}_{k-2}^{(m)}$ is isomorphic to an indecomposable module with the subquotient structure $\mathsf{X}_k \to \mathsf{X}_{k-1}$, and $\mathsf{R}_k^{(m)}/\mathsf{R}_{k-2}^{(m)}$ is isomorphic to $\mathsf{X}_{-k} \to \mathsf{X}_{-k+1}$. These filtrations are used in the next.

As was observed above in Sec. 3.3, all subquotients of T_n in the middle level (those X_k satisfying $k - n = 0 \mod 2$) are divided into two classes – one having only south-west ingoing arrows and south-east outgoing ones, and in opposite, the second class has only south-east ingoing and south-west outgoing arrows. Following this division, we therefore can construct a T_n module as an extension of the modules introduced in (3.5) and (3.6) in the following two ways

$$\mathsf{T}_n \quad \cong \quad \begin{matrix} \mathsf{B}_{n-1}^{(n+1)} \\ \swarrow \\ \mathsf{B}_n^{(n)} \end{matrix} \quad \cong \quad \begin{matrix} \mathsf{R}_{n-1}^{(n+1)} \\ \searrow \\ \mathsf{R}_n^{(n)} \end{matrix} \tag{3.9}$$

where the south-west and south-east arrows depict the action of the positive and negative subalgebras $N_{\mathfrak{q}}^+$ and $N_{\mathfrak{q}}^-$, respectively. This construction of direct summands in the spin-chain decomposition reminds us the open case, where $\mathsf{P}_{1,n}$ module is an extension of a pair of Weyl modules over $U_{\mathfrak{q}}s\ell(2)$.

In order to study decomposition over JTL_N below we now describe all intertwining operators respecting the $U_{\mathfrak{q}}^{\mathrm{odd}}s\ell(2)$ action on the spin-chain. Using the decomposition (3.3), it is enough to describe all homomorphisms among the indecomposable direct summands T_n.

Theorem 3.4.4. *For $n, m \in \mathbb{N}$, we have the equalities*

$$\dim \mathrm{Hom}_{U_{\mathfrak{q}}^{odd}s\ell(2)}(\mathsf{T}_n, \mathsf{T}_m) = \begin{cases} 2, & m = n \pm 1, \\ \min(n, m) + \delta_{n,m}, & m - n = 0 \mod 2, \\ 0, & otherwise. \end{cases} \tag{3.10}$$

The two-dimensional space in the case $m = n+1$ is spanned by homomorphisms $f_{n,n+1}^{\pm}$ with images

$$\mathrm{im}(f_{n,n+1}^+) \cong \mathsf{R}_{n-1}^{(n+1)}, \quad \mathrm{im}(f_{n,n+1}^-) \cong \mathsf{B}_{n-1}^{(n+1)},$$

while the case $m = n-1$ corresponds to maps $f_{n,n-1}^{\pm} \in \mathrm{Hom}(\mathsf{T}_n, \mathsf{T}_{n-1})$ with images

$$\mathrm{im}(f_{n,n-1}^+) \cong \mathsf{R}_{n-1}^{(n-1)}, \quad \mathrm{im}(f_{n,n-1}^-) \cong \mathsf{B}_{n-1}^{(n-1)}.$$

In the case $m - n = 0 \mod 2$, the Hom-space is spanned by homomorphisms with semisimple images.

Proof. We first describe the space $\mathrm{Hom}_{U_{\mathfrak{q}}^{\mathrm{odd}}}(\mathsf{T}_n, \mathsf{T}_m)$ when $n - m = 0 \mod 2$. The subquotient structure of T_n in Fig. 3 makes evident that the only non-trivial intertwining operators from T_n to T_m, with $n - m = 0 \mod 2$ and $n \neq m$, are homomorphisms with images isomorphic to semi-simple submodules in T_m. The corresponding Hom space is spanned by homomorphisms with images isomorphic to X_k, with $k - n = 0 \mod 2$ and $1 - \min(n, m) \leq k \leq \min(n, m) - 1$. In the case $n = m$ we have one more homomorphism given by an idempotent.

Second, it is crucial to note that, for $n \neq m$, non-trivial homomorphisms with images being an indecomposable submodule are only between T_n and $\mathsf{T}_{n\pm 1}$ – this easily follows from (3.9) and the filtrations (3.7) and (3.8). Moreover, as it follows from Fig. 3 above and Fig. 2 in particular, there are only two independent homomorphisms beween T_n and $\mathsf{T}_{n\pm 1}$ – of "positive/south-west" and "negative/south-east" types. A homomorphism $\mathsf{T}_n \to \mathsf{T}_{n+1}$ of the positive type has the kernel isomorphic to $\mathsf{R}_n^{(n)}$ and its image is the submodule $\mathsf{R}_{n-1}^{(n+1)} \subset \mathsf{T}_{n+1}$, where we use again (3.9) and the filtration (3.8); the negative-type homomorphism $\mathsf{T}_n \to \mathsf{T}_{n+1}$ has the image isomorphic to $\mathsf{B}_{n-1}^{(n+1)} \subset \mathsf{T}_{n+1}$, where we use (3.7). To describe the two homomorphisms $\mathsf{T}_n \to \mathsf{T}_{n-1}$, we only note that their kernels are generated by $\mathsf{R}_n^{(n)}$ and the subquotient X_n – for the positive-type homomorphisms, – and by $\mathsf{B}_n^{(n)}$ together with the subquotient X_{-n} – for the negative-type. The images of the last two homomorphisms are isomorphic to $\mathsf{R}_{n-1}^{(n-1)}$ and $\mathsf{B}_{n-1}^{(n-1)}$, respectively. Finally, the subquotient structure (3.9) with the use of the 'Weyl' modules and filtrations in (3.7) and (3.8) show that the two independent homomorphisms from T_n to $\mathsf{T}_{n\pm 1}$ exhaust the $\mathrm{Hom}_{U_{\mathfrak{q}}^{\mathrm{odd}}}$ space. This proves the theorem. $\square$

4 The standard modules over JTL_N

4.1 Generalities

We now go back for a little while to the case of the full affine Temperley–Lieb algebra $T_N^a(m)$. Set $m = \mathfrak{q} + \mathfrak{q}^{-1}$. For generic $\mathfrak{q} \neq 1$, the irreducible representations we shall need are parametrized by two numbers. In terms of diagrams, the first is the number of through-lines, which we denote by $2j$, $j = 0, 1, \ldots, L$, connecting $2j$ sites on the inner and $2j$ sites on the outer boundary of the annulus; the $2j$ sites on the inner boundary we call free or non-contractible. The action of the algebra $T_N^a(m)$ is defined in a natural way on these diagrams, by joining their outer boundary to an inner boundary of a diagram from $T_N^a(m)$, and removing the interior sites. As usual, a closed contractible loop is replaced by m. Whenever the affine diagram thus obtained has a number of through lines less than $2j$, the action is zero. For a given non-zero value of j, it is possible in this action to cyclically permute the free sites: this gives rise to the introduction of a pseudomomentum K (not to be confused with the quantum group generator). Whenever $2j$ through-lines wind counterclockwise around the annulus l times, we unwind them at the price of a factor e^{2ijlK}; similarly, for clockwise winding, the phase is e^{-i2jlK} [7, 8][3]. This action gives rise to a generically irreducible module, which we denote by $\mathcal{W}_{j,e^{2iK}}$. Note that we used a parametrization such that different pairs (j, e^{2iK}) correspond to non-isomorphic modules over the even-rank subalgebra $O_N(m) \subset T_N^a(m)$ introduced in Sec. 2.1. In the parametrization

---

[3] A more pedantic definition due to [5] is the relation

$$\mu = \mu' \circ u_j^n \equiv e^{iKn} \mu',$$

where μ is an affine diagram with $2j$ through lines, u_j is the translational operator acted on through lines by shifting a free site by one, and μ' is so-called standard diagram which has no through lines winding the annulus.

(t, z) chosen in [5], this corresponds to $t = 2j$ and the twist parameter $z^2 = e^{2iK}$.

The dimensions of these modules $\mathcal{W}_{j,e^{2iK}}$ over $T^a_{2L}(m)$ are then given by

$$\hat{d}_j = \binom{2L}{L+j}, \qquad j > 0. \tag{4.1}$$

Note that the numbers do not depend on K (but representations with different e^{iK} are not isomorphic). These generically irreducible modules $\mathcal{W}_{j,e^{2iK}}$ are known also as standard (or cell) $T^a_N(m)$-modules [5].

Keeping $\mathfrak{q}$ generic, degeneracies in the standard modules appear whenever

$$e^{2iK} = \mathfrak{q}^{2j+2k}, \qquad k \text{ is a strictly positive integer.} \tag{4.2}$$

The representation $\mathcal{W}_{j,\mathfrak{q}^{2j+2k}}$ then becomes reducible, and contains a submodule isomorphic to $\mathcal{W}_{j+k,\mathfrak{q}^{2j}}$. The quotient is generically irreducible, with dimension $\hat{d}_j - \hat{d}_{j+k}$. The degeneracy (4.2) is well-known [8, 5] [4]. When $\mathfrak{q}$ is a root of unity, there are infinitely many solutions to the equation (4.2), leading to a complex pattern of degeneracies to which we turn below.

The case $j = 0$ is a bit special. There is no pseudomomentum, but representations are still characterized by another parameter, related with the weight given to non contractible loops. Parametrizing this weight as $z + z^{-1}$, the corresponding standard module of $T^a_{2L}(m)$ is denoted $\mathcal{W}_{0,z^2}$.

We now specialize to the Jones–Temperley–Lieb algebra $JTL_N(m)$ defined in Sec. 2.1. In this case, the rule that winding through-lines can simply be unwound means that the pseudomomentum must satisfy $jK \equiv 0 \bmod \pi$ [12]. All possible values of the parameter $z^2 = e^{2iK}$ are thus j-th roots of unity ($z^{2j} = 1$, [10]). The kernel of the homomorphism ψ in (2.3) (and the ideal in $T^a_N(m)$ generated by $u^N - 1$, in particular) acts trivially on these modules if $j > 0$. In what follows, we will thus use the same notation $\mathcal{W}_{j,z^2}$, with $j > 0$, for the standard $JTL_N(m)$-modules. We note that two standard JTL_N-modules having only different signs in the z parameter are isomorphic.

If $j = 0$, requiring the weight of the non contractible loops to be m as well leads to the $T^a_N(m)$-module $\mathcal{W}_{0,\mathfrak{q}^2}$ which is reducible even for generic $\mathfrak{q}$ – it contains a submodule isomorphic to $\mathcal{W}_{1,1}$. Meanwhile, on the standard module $\mathcal{W}_{0,\mathfrak{q}^2}$ the kernel of the homomorphism ψ is non-trivial: the standard module over $JTL_N(m)$ for $j = 0$ is obtained precisely by taking the quotient $\mathcal{W}_{0,\mathfrak{q}^2}/\mathcal{W}_{1,1}$ as in [5]. This module is now simple for generic $\mathfrak{q}$, has the dimension $\binom{2L}{L} - \binom{2L}{L-1}$ and is denoted by $\overline{\mathcal{W}}_{0,\mathfrak{q}^2}$.

In what follows we use the representation theory [5] of T^a_N in order to describe the subquotient structure of JTL_N-standard modules. For this it is convenient to use a variant of JTL_N, which is also embedded in $T^a_N(m)$. This variant (dubbed here augmented) is the finite-dimensional algebra $JTL_N^{(au)}(m)$, isomorphic to $JTL_N(m)$ except for the ideal without through-lines. In this ideal, the algebra $JTL_N^{(au)}(m)$ differs from $JTL_N(m)$ in that connections within the points on the inner or outer annulus, which are topologically different are treated as different. Recall that in $JTL_N(m)$, diagrams in the ideal with no through lines can be chosen to be planar (they can be drawn in a box without

---

[4]Note that the twist term in [17], which was denoted there q^{2t}, reads in these notations as e^{2iK}. It corresponds to z^2 in the Graham–Lehrer work [5], and to the parameter x in the work of Martin–Saleur [8]. The case where $k = 1$ is special, and related with braid translation of the blob algebra theory. We note that in the JTL_N case, $2j$ lines going around the cylinder pick up a phase $e^{i2jK} = 1$. In [8], this corresponds to $\alpha_h = x^h = 1$.

crossings), and are in bijection with ordinary TL_N-diagrams. This distinction leads to the standard $JTL_N^{(au)}$-module $\mathcal{W}_{0,\mathfrak{q}^2}$ of dimension $\binom{2L}{L}$.

Several results can easily be established following [5] when $\mathfrak{q} = i$, to which we restrict for now. We note that the dimension of the sector of value $S^z = j$ or $S^z = -j$ (including $j = 0$) in the spin chain coincides with the dimension $\hat{d}_j$ of the standard module $\mathcal{W}_{j,e^{2iK}}$ over the augmented algebra $JTL_N^{(au)}$. For $\mathfrak{q} = i$, these spin-chain sectors provide highly reducible representations of the Jones–Temperley–Lieb algebra JTL_N closely related (but non-isomorphic) to the standard modules. By the discussion of the correspondence [1] between the XX and the $g\ell(1|1)$ spin-chains, we see that these representations occur at pseudomomentum satisfying $e^{2iK} = (-1)^{j+1}$. Before describing indecomposables appearing in the spin-chain we first discuss more the standard ones with this pseudomomentum.

4.2   The standard modules at $\mathfrak{q} = i$

We first describe modules over the algebra T_N^a, containing the generator u. The structure of the standard T_N^a-modules at $\mathfrak{q} = i$ can be inferred from [5]. For a standard module $\mathcal{W}_{j,(-1)^{j+1}}$ with $2j > 0$ through lines, we deduce the subquotient structure using two Graham–Lehrer's theorems, Thm. 3.4 and proof of Thm. 5.1 in [5]. A crucial fact is that the space of homomorphisms

$$\mathrm{Hom}_{T_N^a}(\mathcal{W}_{j,(-1)^{j+1}}, \mathcal{W}_{j-1,(-1)^j}) \cong \mathbb{C}, \qquad 1 \leq j \leq L. \tag{4.3}$$

between the standard T_N^a-modules is one-dimensional and the homomorphisms are injective. The dimensions of simple modules $\mathcal{L}_{j,(-1)^{j+1}}$ happen to be the same as those in the open case, and given by

$$\widehat{d}^0_{j,(-1)^{j+1}} = d^0_j = \sum_{j' \geq j}(-1)^{j'-j}d_{j'} \qquad \text{with} \qquad d_j = \binom{2L}{L+j} - \binom{2L}{L+j+1}.$$

One can show the equivalent formula

$$\widehat{d}^0_{j,(-1)^{j+1}} = \binom{2L-2}{L-j} - \binom{2L-2}{L-j-2}.$$

Our final result for the standard T_N^a-modules is given on the left side of Figs. 4 and 5 where each node corresponds to a simple subquotient. In the case $j = 0$, we have no top subquotient because $d^0_0 = 0$. We denote the dimension of a simple subquotient $\mathcal{L}_{k,(-1)^{k+1}}$ in the round brackets (with the twist parameter $z = \pm\sqrt{(-1)^{k+1}}$ for each node $(d^0_k)_\pm$, with $j \leq k \leq L$, to be assumed). For simplicity, we use in what follows the round-brackets notation for simple subquotients. We will also denote the Graham–Lehrer's parameter $z = \pm\sqrt{z^2}$ by the subscript $\pm$ distinguishing non-isomorphic simple T_N^a-subquotients. Restricting to the subalgebra $JTL_N^{(au)}$, subquotients $(d^0_k)_\pm$ are isomorphic and we leave any subscripts in modules over JTL_N.

We now turn to the description of standard modules over the subalgebra $JTL_N^{(au)}$.

Proposition 4.2.1. *The subquotient structures for the standard JTL_N-modules $\mathcal{W}_{j,(-1)^{j+1}}$, with $j > 0$, and for the standard $JTL_N^{(au)}$-module $\mathcal{W}_{0,-1}$ are given on the right in Figs. 4 and 5, respectively.*

Proof. The proof consists of two parts **1.** and **2.** The first one consider the case $j = 0$ and it is then used in **2.** to deduce the structure for $j > 0$.
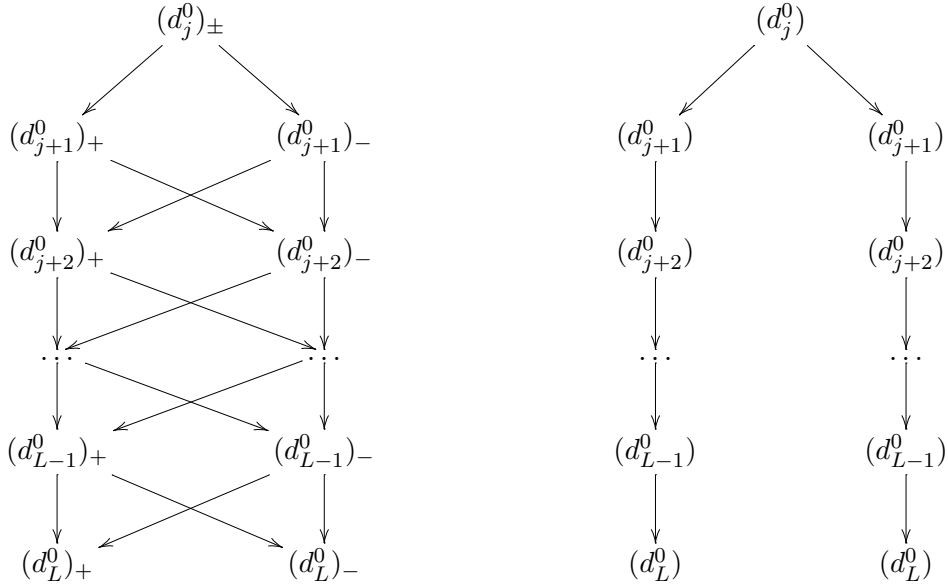
Figure 4: The structure of the standard modules $\mathcal{W}_{j,(-1)^{j+1}}$ with $2j > 0$ through lines at $\mathfrak{q} = i$. We set $\mathcal{L}_{j,(-1)^{j+1}} \equiv (d_j^0)$. The module on the left is over T_N^a and on the right is the restriction to the subalgebra $JTL_N^{(au)}$. The twist parameter $z = \pm\sqrt{(-1)^{k+1}}$ for each node $(d_k^0)_\pm$ is assumed.

1. For the standard $JTL_N^{(au)}$-module $\mathcal{W}_{0,-1}$ without through lines, the subquotient structure degenerates into a direct sum of two non-isomorphic indecomposable modules each consisting of affine diagrams of even or odd rank [5]. These two summands are of chain type and presented on Fig. 5, the right diagram. Black arrows represent the action of the subalgebra TL_N – open Temperley–Lieb algebra – generated by e_j, with $1 \leq j \leq N - 1$, and red arrows indicate an action of the last generator e_N that mixes the direct summands over TL_N. The left direct summand over JTL_N is spanned by affine diagrams μ of even rank $0 \leq |\mu| \leq L$, the right summand – by odd-rank diagrams. This picture easily follows from the filtration [5] of $\mathcal{W}_{0,-1}$ by the standard TL_N-modules.

We note next that the translation operator u ($ue_ku^{-1} = e_{k+1}$) mixes even affine diagrams with odd ones. The corresponding standard module for $j = 0$ with respect to the bigger algebra T_N^a containing the element u has the subquotient structure given in Fig. 5 on the left side. By selecting a node further down in the ladder, and truncating all that is at its level or above, one can obtain as well the structure of all the other standard modules over T_N^a presented on the left side in Fig. 4, using (4.3) and injectivity of the homomorphisms. We recall that the subscript $\pm$ distinguishes non-isomorphic T_N^a-irreducibles, and that there are actually two standard T_N^a-modules, with the top $(d_j^0)_\pm$, corresponding to the notation $\mathcal{W}_{j,(-1)^{j+1}}$.

2. Restricting to $JTL_N^{(au)}$, the simple modules $(d_j^0)_+$ and $(d_j^0)_-$ as well as their standard modules are isomorphic as modules over $JTL_N^{(au)}$ and we thus have the isomorphism of vector spaces

$$\text{Hom}_{JTL_N^{(au)}}(\mathcal{W}_{j,(-1)^{j+1}}, \mathcal{W}_{j-1,(-1)^j}) \cong \mathbb{C}^2, \qquad 1 \leq j \leq L. \tag{4.4}$$

Using this isomorphism, we now show that the 'diagonal' arrows connecting the left and right strands in T_N^a-modules are absent in the corresponding $JTL_N^{(au)}$-modules, i.e., they represent actually the
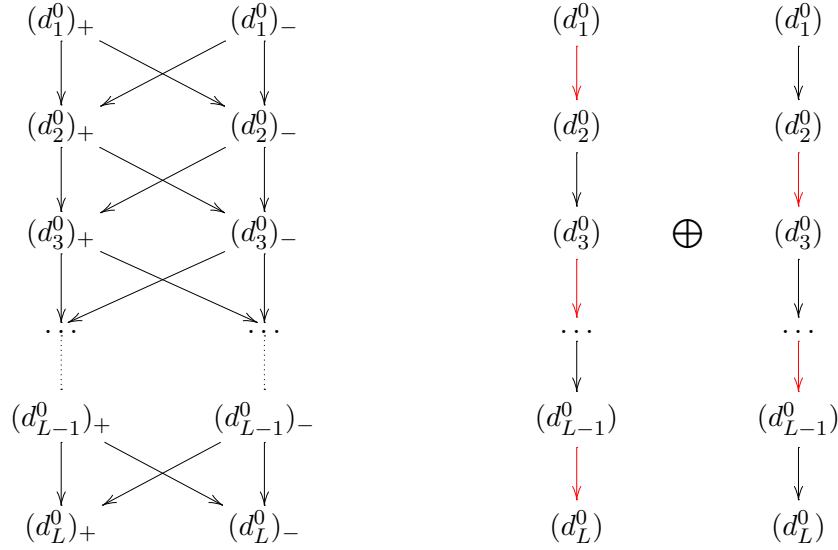
Figure 5: The structure of the standard T_N^a-module $\mathcal{W}_{0,-1}$ at $\mathfrak{q} = i$ is on the left side, while the corresponding standard $JTL_N^{(au)}$-module $\mathcal{W}_{0,-1}$ (but not over JTL_N) is given on the right side. For the last $\mathcal{W}_{0,-1}$, we show a decomposition on standard TL_N-modules using black and red arrows. Nodes connected by black arrows constitute a standard module over TL_N while red arrows indicate an action of the last generator e_N that mixes the direct summands over TL_N.
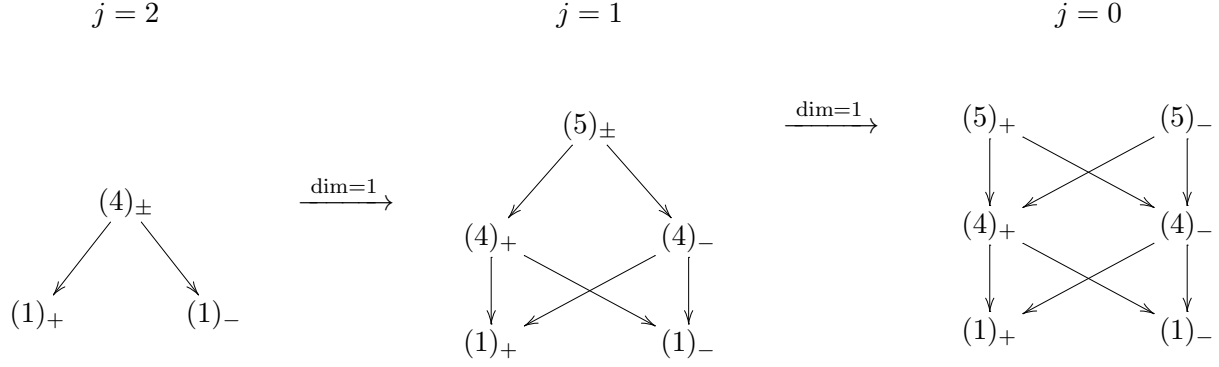
action of the ideal in T_N^a generated by the element u. We begin with studying homomorphisms from $\mathcal{W}_{1,1}$ to $\mathcal{W}_{0,-1}$. We remind that the last module is a direct sum of two indecomposables each consisting of affine diagrams of even or odd rank as in Fig. 5 on the right side, and each having the same top (d_1^0) as the $\mathcal{W}_{1,1}$. Therefore, a basis in (4.4) for $j = 1$ can be chosen as two homomorphisms with the image isomorphic to the left or right direct summand in $\mathcal{W}_{0,-1}$ in Fig. 5. This means the kernel of any of these homomorphisms contains either the submodule $(d_2^0) \to (d_3^0) \to \dots$ – the chain started with the red arrow – or the one started with the black arrow. The kernels are submodules over $JTL_N^{(au)}$ and we thus can choose a basis in $\mathcal{W}_{1,1}$ such as there are no arrows (with respect to the action of $JTL_N^{(au)}$) mixing these submodules. We proceed in the same way for $j > 1$. This finally gives the diagrams for the modules $\mathcal{W}_{j,(-1)^{j+1}}$ over $JTL_N^{(au)}$ in Fig. 4 on the right. In these diagrams, we could also indicate the action of e_N by red arrows connecting standard TL_N-modules in a decomposition over the subalgebra generated by e_j, with $1 \le j \le N-1$, as in Fig. 5, but the diagrams in such a basis would probably contain some 'diagonal' arrows connecting the left and right strands in $\mathcal{W}_{j,(-1)^{j+1}}$.

By the definition of $JTL_N^{(au)}$ algebra given above in Sec. 4.1, the JTL_N-modules $\mathcal{W}_{j,(-1)^{j+1}}$ for $j > 0$ have the same subquotient structure as in the right diagram in Fig. 4. This finishes the proof. $\square$

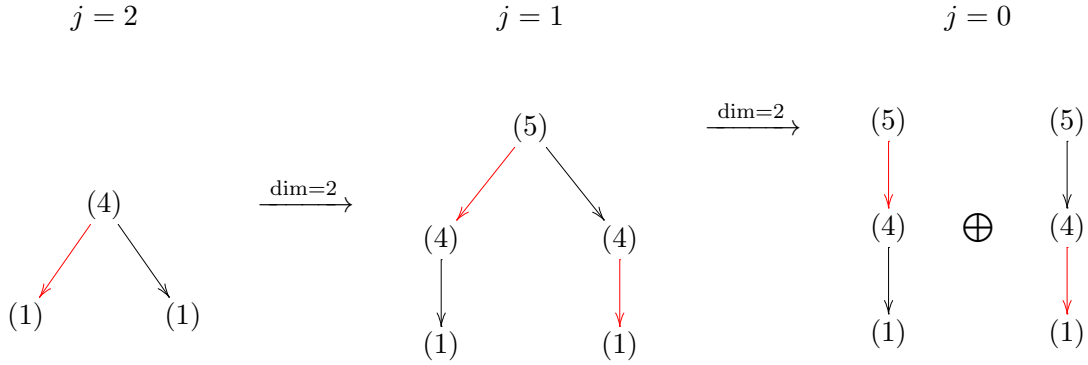We finally give some explicit examples.

Example 4.2.2. For $L = 3$ or $N = 6$, we have the following diagrams for subquotient structure of

the standard T_N^a-modules $\mathcal{W}_{j,(-1)^{j+1}}$.

$$j = 2 \qquad\qquad j = 1 \qquad\qquad j = 0$$

$$(5)_\pm$$
$$\xrightarrow{\text{dim=1}} \qquad (5)_+ \qquad\qquad (5)_-$$

$$(4)_\pm \qquad \xrightarrow{\text{dim=1}} \qquad (4)_+ \qquad (4)_- \qquad\qquad (4)_+ \qquad (4)_-$$

$$(1)_+ \qquad (1)_- \qquad\qquad (1)_+ \qquad (1)_- \qquad\qquad (1)_+ \qquad (1)_-$$

with indicated injective homomorphisms between them. We also show the dimension of the spaces of homomorphisms between the standard T_N^a-modules in the figure.

The diagrams for subquotient structure of the modules $\mathcal{W}_{j,(-1)^{j+1}}$ over $JTL_N^{(au)}$ are

$$j = 2 \qquad\qquad j = 1 \qquad\qquad j = 0$$

$$(5) \qquad \xrightarrow{\text{dim=2}} \qquad (5) \qquad\qquad (5)$$

$$(4) \qquad \xrightarrow{\text{dim=2}} \qquad (4) \qquad (4) \qquad\qquad (4) \quad \oplus \quad (4)$$

$$(1) \qquad (1) \qquad\qquad (1) \qquad (1) \qquad\qquad (1) \qquad (1)$$

where we also show the filtration [5] of the standatd $JTL_N^{(au)}$-modules by the standard TL_N-modules and the red arrows represent the action of the generator e_6. They are the same diagrams as in Fig. 4 and Fig. 5 but truncated for $L = 3$. The most left diagram is for the sector with $2j = 4$ through lines ($\hat{d}_2 = 6$), the central one is spanned by affine diagrams with 2 through lines ($\hat{d}_1 = 15$), and the right most diagram has no through lines ($j = 0$, $\hat{d}_0 = 20$). The two invariants in $\mathcal{W}_{0,-1}$ are given explicitly by

$$\text{inv}_1 = \sum_{j=1}^{3} u^{2j}\left(\cup\, \cup\!\cup - \cup\!\cup\!\cup \right) \qquad \text{inv}_2 = u(\text{inv}_1).$$

The two T_N^a-invariants $(1)_\pm$ on the diagram above are spanned by $\text{inv}_1 \pm \text{inv}_2$, respectively.

5 The spin-chain decomposition over JTL_N

It turns out that the structure of the modules present in the $g\ell(1|1)$ spin chain is closely related to the standard modules discussed above. First, we give some results about extensions between ('glueings' of) simple modules and give explicit examples. Then, we construct "zig-zag" indecomposable JTL_N-modules that play the role of the standard modules for TL_N in the spin-chain decomposition, i.e., indecomposable direct summands over JTL_N in the spin-chain are gluings of two such zig-zag modules. Finally, we use these modules to describe the subquotient structure of spin-chain modules over JTL_N and state a bimodule structure over the pair $\left(JTL_N, U_q^{\text{odd}} s\ell(2)\right)$.

5.1 Extensions between simple JTL_N-modules

As a consequence of the subquotient structure for the standard JTL_N-modules studied in the previous section, we can state the following result.

Lemma 5.1.1. *The dimension of the group of first extensions between simple $JTL_N(0)$-modules $\mathcal{L}_{n,(-1)^{n+1}}$ and $\mathcal{L}_{m,(-1)^{m+1}}$, for $n = m \pm 1$, is not less than 2.*

We will formulate below a conjecture proposing that the dimension of the extension group is precisely two – it is motivated by our analysis of projective and tilting JTL_N-modules, which will be published elsewhere – but Lem. 5.1.1 is enough for our purposes in studying spin-chain decompositions.

In what follows, we use a notation for basis elements denoted by $\boldsymbol{x}_\pm$ and $\boldsymbol{y}_\pm$ that span a two-dimensional subspace in the first extension groups $\mathrm{Ext}^1_{JTL_N}\big(\mathcal{L}_{n,(-1)^{n+1}}, \mathcal{L}_{n\pm1,(-1)^n}\big)$ from Lem. 5.1.1. The basis element $\boldsymbol{x}_\pm$ is chosen to represent an extension corresponding to the action of the open Temperley–Lieb subalgebra TL_N generated by e_j, with $1 \leq j \leq N-1$, and it is depicted by an arrow connecting two simple subquotients $\mathcal{L}_{n,(-1)^{n+1}}$ and $\mathcal{L}_{n\pm1,(-1)^n}$. The second extension $\boldsymbol{y}_\pm$ corresponds to the action of the subalgebra $uTL_Nu^{-1} \subset JTL_N$ isomorphic to TL_N and containing the generator e_N and it is depicted by a second arrow connecting the same pair of subquotients as in the diagram

$$\tag{5.1}$$

where the coefficients $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, and we set $\mathcal{L}_{0,-1} \equiv 0$. We note that different elements in the intersection of the two subalgebras TL_N and uTL_Nu^{-1} can actually map to different linear combinations of the simple submodules; this is not shown explicitly on the diagram. We do not give a rigorous proof of existence of these two extensions but we nevertheless refer a reader to our discussion in the proof of Prop. 4.2.1. In the proof, we give a decomposition of standard JTL_N-modules on standard modules over the subalgebra TL_N generated by e_j, with $1 \leq j \leq N-1$, and show the action of e_N connecting the direct summands.
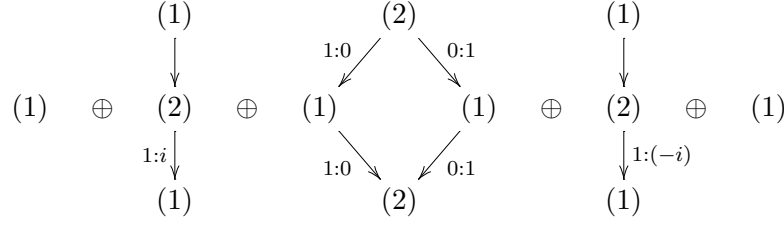
Taking all possible quotients of the module in (5.1) by a submodule isomorphic to the direct sum $\mathcal{L}_{n-1,(-1)^n} \oplus \mathcal{L}_{n+1,(-1)^n}$, we obtain a family of indecomposable JTL_N-modules with the subquotient structure

$$\mathcal{L}_{n-1,(-1)^n} \underset{(\alpha:\beta)\boldsymbol{y}_-}{\overset{\boldsymbol{x}_-}{\rightleftarrows}} \mathcal{L}_{n,(-1)^{n+1}} \underset{(\delta:\gamma)\boldsymbol{y}_+}{\overset{\boldsymbol{x}_+}{\rightleftarrows}} \mathcal{L}_{n+1,(-1)^n} \tag{5.2}$$

and parametrized by two points $x_0 = \alpha : \beta$ and $y_0 = \delta : \gamma$ on a complex projective line, $x_0, y_0 \in \mathbb{CP}^1$. These modules are denoted by $\mathcal{M}_n^{(1)}(x_0, y_0)$ and they will appear below in spin-chain decompositions. To simplify notations, we will use below only single arrows with specified parameters on them. Before going to decomposition, we first give an example at $N = 4$ where parameters on $\mathbb{CP}^1$ appear.

5.1.2 Example for $N = 4$

The decomposition of the full spin-chain on $N = 4$ sites with respect to JTL_N action is given by the direct sum, where we set for simple subquotients $\mathcal{L}_{1,1} = (2)$ and $\mathcal{L}_{2,-1} = (1)$,

$$
\begin{array}{ccccccccc}
& & (1) & & (2) & & (1) & & \\
& & \downarrow & {}^{1:0}\!\nearrow \ \searrow^{0:1} & & \downarrow & & \\
(1) & \oplus & (2) & \oplus \quad (1) \qquad\qquad (1) & \oplus & (2) & \oplus & (1) \\
& {}_{1:i}\!\downarrow & & {}^{1:0}\!\searrow \ \nearrow^{0:1} & & \downarrow^{1:(-i)} & & \\
& & (1) & & (2) & & (1) & &
\end{array}
$$

where the left-most direct summand is at $S^z = 2$, the second is at $S^z = 1$, etc., see also a general decomposition in (5.4) below. The only isomorphic modules are the two invariants depicted by (1) and mapped to each other by e^2 and f^2. The two modules at $S^z = \pm 1$ are non-isomorphic – they differ by the points on $\mathbb{CP}^1$ indicated as $(1 : \pm i)$ on the lower parts of their diagrams; in other words, the arrow from (2) to (1) in the sector $S^z = 1$, on the left side, corresponds to the extension $\boldsymbol{x}_+ + i\boldsymbol{y}_+$ while the submodule (2) $\to$ (1) at $S^z = -1$ corresponds to the extension $\boldsymbol{x}_+ - i\boldsymbol{y}_+$. The basis extensions $\boldsymbol{x}_+$ and $\boldsymbol{y}_+$ are introduced before (5.1) and here they simply mean that e_1 maps from the two-dimensional subquotient (2) to the one-dimensional (1) with the coefficient 1 in an appropriate basis in the submodule (1) while e_3 maps with the coefficient $\pm i$, in the same basis of course.

5.2 Spin-chain modules over JTL_N

We first recall a decomposition of $\mathcal{H}_N$ over the TL_N, the open case [3].

$$
\mathcal{H}_N|_{TL_N} = \bigoplus_{j=1}^{L} \mathcal{P}_j \boxtimes \mathsf{X}_{1,j} \oplus \mathcal{W}_L \boxtimes \mathsf{X}_{1,L+1}, \tag{5.3}
$$

with "multiplicities" $\mathsf{X}_{1,j}$ in front of indecomposable direct summands $\mathcal{P}_j$ being simple j-dimnesional modules over $U_{\mathsf{q}}s\ell(2)$ defined in (B1). We use the notations $\mathcal{P}_j$ and $\mathcal{W}_j$ for projective and standard TL_N-modules, respectively. The standard module $\mathcal{W}_L$ is the trivial representation (1); the standard modules with $1 \le j < L$ have structure of simple subquotients as $\mathcal{W}_j : (d_j^0) \to (d_{j+1}^0)$, and $\mathcal{W}_0$ is the simple module (d_1^0). The projectives $\mathcal{P}_j$ are self-conjugate and described by the diagram $\mathcal{W}_j \to \mathcal{W}_{j-1}$.

In general in the periodic case, the JTL_N action commutes with $S^z = 2\mathsf{h}$ and we have a decomposition of the full spin-chain over JTL_N on $N = 2L$ sites as

$$
\mathcal{H}_N|_{JTL_N} = \bigoplus_{j=-L+1}^{L-1} \widehat{\mathcal{P}}_j \boxtimes \mathsf{X}_j \oplus \mathcal{W}_{L,(-1)^{L-1}} \boxtimes \mathsf{X}_L, \tag{5.4}
$$

where $\widehat{\mathcal{P}}_j$ denotes a unique indecomposable module in the sector $S^z = j$ which we call *the spin-chain JTL_N-module*, and X_j is the one-dimensional and X_L is the two-dimensional simple $\mathfrak{Z}_{JTL}$-modules (the representation theory of the centralizer $\mathfrak{Z}_{JTL}$ is described in Sec. 3). We also set $\widehat{\mathcal{P}}_L \equiv \mathcal{W}_{L,(-1)^{L-1}}$ in what follows.

Subquotient structure of the spin-chain JTL_N-modules $\widehat{\mathcal{P}}_j$ can be obtained using the centralizing property with the $\mathfrak{Z}_{JTL}$ algebra which is essentially the representation $\rho_{g\ell}$ of $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$. The decomposition of the spin-chain over $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$ together with the subquotient structure of each indecomposable summand T_n is given in (3.3).

In a double-centralizing situation, it would be sufficient to use Thm. 3.4.4 describing all intertwining operators between indecomposable T_n and T_m modules over the JTL-centralizer in order to reconstruct all arrows for the subquotient structure depicting the JTL_N action. The double centralizing property is obvious in a semisimple case but it is not evident for the representation we consider here. We show below that the centralizer of $\mathfrak{Z}_{JTL}$ is actually a larger algebra containing the representation $\pi_{g\ell}$ of JTL_N. Our strategy is thus requires a revision of the one described in Sec. 2.3, where the double-centralizing property was assumed for simplicity, and consists of the following steps. We propose first a subquotient structure for the spin-chain JTL_N-modules $\widehat{\mathcal{P}}_j$ and study then all homomorphisms between these modules and finally identify corresponding intertwining operators with PBW basis elements of $U_q^{\text{odd}} s\ell(2)$ given in Def. 2.4.1 and that are represented faithfully. At this step, we realize only a sufficient condition on the module structure to have the centralizer $\mathfrak{Z}_{JTL}$. To show that the subquotient structure indeed corresponds to the JTL_N action, we do a further analysis involving fermionic expressions for generators of $\pi_{g\ell}(JTL_N)$ computed in [1] and Thm. 3.4.4 giving a subquotient structure for $\mathcal{H}_N$ as a module over the centralizer of $\mathfrak{Z}_{JTL}$.

Before going to a decomposition for any even N, we first give our results for $N = 8$ in order to give more experience with modules and arrows.

5.2.1 Decomposition for $N = 8$

Following the strategy described above and using the decomposition over $U_q^{\text{odd}} s\ell(2)$ for $N = 8$ in Fig. 2, we find a decomposition of the full spin-chain over JTL_8. The decomposition is the direct sum (5.4) with each indecomposable summand $\widehat{\mathcal{P}}_j$ given from left to right corresponding to the decreasing value of $-4 \leq j \leq 4$ in the sum


$$(5.5)$$

where we set $\mathcal{L}_{1,1} = (14)'$, $\mathcal{L}_{2,-1} = (14)$, $\mathcal{L}_{3,1} = (6)$, and $\mathcal{L}_{4,-1} = (1)$ indicating dimensions of simple subquotents in the round brackets. In the diagrams, we introduce arrows of two types as in Fig. 5 – black arrows show the action of the subalgebra $TL_N \subset JTL_N$ generated by e_j, $1 \leq j \leq N-1$ (this fixes a basis in each sector S^z, up to a basis in each TL_N-summand), and red arrows show an action of the last generator e_N that mixes the direct summands over TL_N (the e_N can act non-trivially also along the black arrows in such a basis); ignoring the red arrows gives a decomposition over TL_N for each sector at $-3 \leq S^z \leq 3$ which is obtained from (5.3) by the restriction on the chosen value of S^z. We depict the arrows without their projective-line parameters introduced in Sec. 5.1 for brevity.

We describe now intertwining operators respecting the subquotient structure proposed in (5.5). We note first that the only isomorphic modules in (5.5) are the two invariants depicted by (1) and

connected by the action of e^4 and f^4, otherwise we would have an intertwining operator not from $\mathfrak{Z}_{JTL}$. From the relations (2.12) and the PBW basis given in Def. 2.4.1, we see that each space $\mathrm{Hom}_{JTL_N}(\widehat{\mathcal{P}}_j, \widehat{\mathcal{P}}_k)$ should be one-dimensional whenever $j - k = 1 \bmod 2$ and spanned by $\mathsf{F}_{(j-k-1)/2}$, if $j > k$, and $\mathsf{E}_{(k-j-1)/2}$, if $j < k$, times appropriate polynomial in h projecting onto $\widehat{\mathcal{P}}_j$. We denote the projector by $p_j(\mathsf{h})$ and it is defined as

$$p_j(\mathsf{h}) = \prod_{n=-N/2;\, n \neq j}^{n=N/2} (2\mathsf{h} - n). \tag{5.6}$$

In order to see the corresponding homomorphisms between the direct summands in (5.5), we consider the case $j = 0$ (the right diagram) and $k = 1$ (the left one)
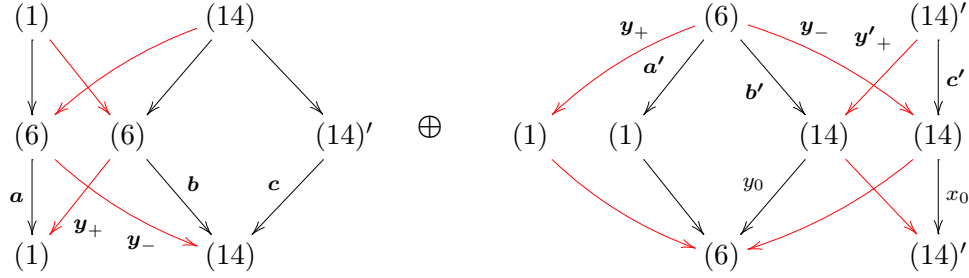
where we mark arrows by corresponding representatives from the first-extensions groups for JTL_N, see Lem. 5.1.1 and a discussion below the lemma (we do not suppose that extensions marked by first latin letters, like $\boldsymbol{a}$, are linear combinations of $\boldsymbol{x}_+$ and $\boldsymbol{y}_+$ introduced in Sec. 5.1, and use only a lower bound stated in Lem 5.1.1.) The homomorphism mapping the right diagram ($k = 0$) to the left one ($j = 1$) corresponds to $\mathsf{E}\, p_0(\mathsf{h})$ and has the image isomorphic to a submodule with the subquotient structure $(1) \leftarrow (6) \rightarrow (14) \leftarrow (14)'$ where the subquotient (6) is in a linear combination of the pair of (6)'s in the left diagram. The kernel of the homomorphism is generated from a linear combination of the two (1)'s and a linear combination of the two (14)'s in the right diagram. The linear combinations can in principle be computed using the decomposition over TL_N and $U_{\mathsf{q}}s\ell(2)$, and the action in projective $U_{\mathsf{q}}s\ell(2)$-modules from App. B but we do not need it. What we get are linear relations among the extensions $\boldsymbol{a}' = \alpha \boldsymbol{a} + \gamma \boldsymbol{y}_+$ and $\boldsymbol{b}' = \beta \boldsymbol{b} + \delta \boldsymbol{y}_-$, and similarly for $\boldsymbol{c}'$, where α, β, γ, δ are some complex numbers. We see also that there no more homomorphisms from the right diagram to the left: an image isomorphic to $(1) \leftarrow (6)$ is not possible because $\boldsymbol{y}_-$ and $\boldsymbol{b}$ in the left diagram are linearly independent, an image isomorphic to $(1) \leftarrow (6) \rightarrow (14)$ is not possible too because this would require the top $(14)'$ in the right diagram to be in the kernel and the both (14) are thus in the kernel too. Similar analysis can be made for all other pairs (j, k) and it shows that the decomposition (5.5) has the algebra of intertwining operators isomorphic to $\mathfrak{Z}_{JTL}$.

At this moment, we have only shown that a sufficient condition on the module structure to have the centralizer $\mathfrak{Z}_{JTL}$ holds. We can not have more arrows in the diagrams, see a general proof after Thm. 5.5. Next, we show that removing at least one red arrow in the decomposition (5.5) results in an enlarged endomorphism algebra. Let's suppose that the arrow connecting the top $(d_1^0) = (14)'$ with $(d_2^0) = (14)$ and marked by $\boldsymbol{y}'_+$ in the right diagram, for $S^z = 0$, is absent. We note the self-conjugacy $(e_j^\dagger = e_j)$ of the JTL_N-representation $\pi_{g\ell}$ in (2.4) which implies that $\widehat{\mathcal{P}}_0^* \cong \widehat{\mathcal{P}}_0$. Therefore, another arrow mapping from the same subquotient (d_2^0) to (d_1^0) in the bottom should be also absent. Then, there exists an extra homomorphism from $S^z = 0$ to $S^z = 1$ with the image $(1) \leftarrow (6) \rightarrow (14)$ because
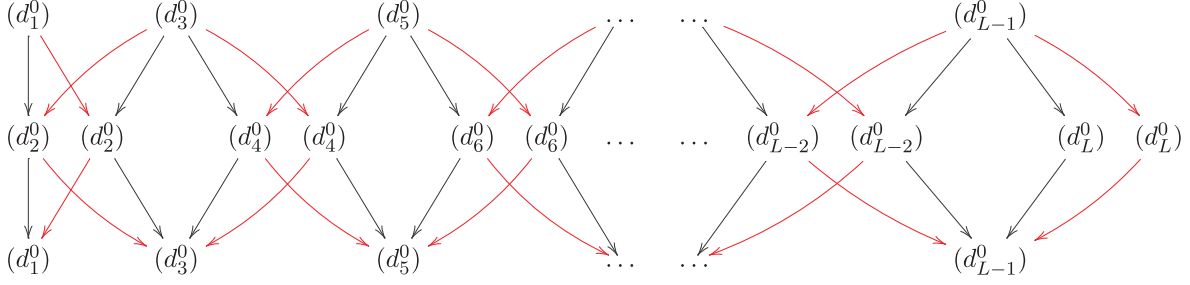
Figure 6: The subquotient structure for the JTL_N-module $\widehat{\mathcal{P}}_0$ corresponding to the sector $S^z = 0$ and $L = 0 \bmod 2$. For odd L, $\widehat{\mathcal{P}}_0$ is obtained by changing $L \to L+1$ and then removing nodes (d^0_{L+1}).

in this case we can take the top (d^0_1) to be in the kernel and we can still embed the top $(d^0_3) = (6)$ into a linear combination of the two (d^0_3)'s in the middle level of the left diagram due to the linear relation between $\boldsymbol{b}'$ and $\boldsymbol{b}$, $\boldsymbol{y}_+$ stated above. The extra homomorphism is not from $\boldsymbol{\mathfrak{Z}}_{JTL}$ and we thus get a contradiction to Thm. 2.4.3. We could in addition suppose that there is no the arrow marked by $\boldsymbol{y}_-$ in the left diagram, and the arrow from the top (d^0_2) to (d^0_3) should be thus absent too. Then, the extra homomorphism from $S^z = 0$ to $S^z = 1$ does not exist in general but we get an extra homomorphism from $S^z = 1$ to $S^z = 2$ which is also not from $\boldsymbol{\mathfrak{Z}}_{JTL}$. The analysis can be repeated for any direct summand in the decomposition (5.5).

So far, we considered consequences of absence of arrows from a top (d^0_j) to (d^0_{j+1}). To say what happens if we suppose absence of an arrow mapping a top (d^0_j) to (d^0_{j-1}) requires more delicate analysis of the extensions. Let's suppose that the arrow connecting the top $(d^0_3) = (6)$ with $(d^0_2) = (14)$ and marked by $\boldsymbol{y}_-$ in the right diagram, for $S^z = 0$, is absent. We also mark the right-most arrow from the top (d^0_3) to the right node (d^0_2) in the diagram for $S^z = 2$ (it is the third summand in (5.5)) by a corresponding extension $\boldsymbol{d}$. Then, mapping by $\mathsf{F}_0 = \mathsf{F}$ the module $\widehat{\mathcal{P}}_0$ to $\widehat{\mathcal{P}}_{-1}$, and by F_1 the $\widehat{\mathcal{P}}_2$ to $\widehat{\mathcal{P}}_{-1}$, we get that $\boldsymbol{b}'$ is proportional to the $\boldsymbol{d}$ because the two operators map the two top (d^0_3)'s to *the same* linear combination of two (d^0_3)'s in the middle level of $\widehat{\mathcal{P}}_{-1}$. On the other hand, mapping $\widehat{\mathcal{P}}_2$ to $\widehat{\mathcal{P}}_1$ by F and $\widehat{\mathcal{P}}_0$ to $\widehat{\mathcal{P}}_1$ by E, we get that the two extensions $\boldsymbol{b}'$ and $\boldsymbol{d}$ should be linearly independent. This contradiction can be solved only by the presence of the arrow marked by $\boldsymbol{y}_-$ in the diagram for $\widehat{\mathcal{P}}_0$.

We finally conclude that JTL_N action mixes the direct summands over TL_N in each sector into one indecomposable module in the way described just above and in (5.5).

5.3 The spin-chain decomposition over JTL_N: the general case

We find finally the spin-chain decomposition over JTL_N for any even N number of sites. Following the examples given above, we see that JTL_N action mixes all the projective modules over the subalgebra TL_N in each subspace with $S^z = j$ into one indecomposable module $\widehat{\mathcal{P}}_j$. Using the decomposition (5.3) over the TL_N subalgebra, we propose the subquotient structure for $\widehat{\mathcal{P}}_0$ given in Fig. 6 for $L = 0 \bmod 2$ ($2L = N$), and we set as usually $\mathcal{L}_{j,(-1)^{j-1}} = (d^0_j)$. Here, again as in the example for $N = 8$, we see that removing all red arrows gives the decomposition over the open Temperley–Lieb $TL_N \subset JTL_N$ into a direct sum of its projective and trivial modules.

We note that the diagram for $\widehat{\mathcal{P}}_0$ can be depicted in a more familiar way as a module with the 'two-strands' subquotient structure (of "Feigin–Fuchs" type) presented on the left of Fig. 7, where we do not use colors and it is supposed that arrows connecting isomorphic subquotients correspond to
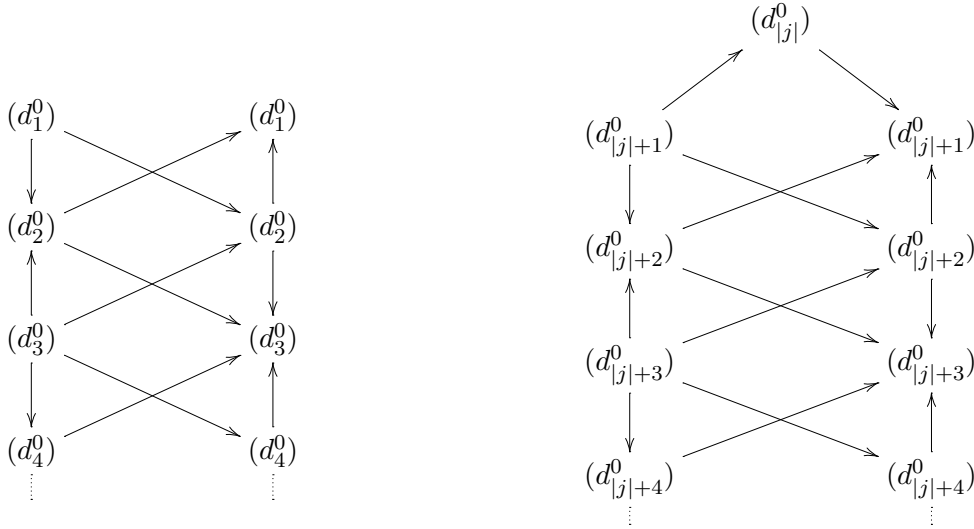
Figure 7: The two-strands structure of the spin-chain JTL_N-modules $\widehat{\mathcal{P}}_j$ for $j = 0$ on the left and $j \neq 0$ on the right side. The towers are ended by the pair of (d_L^0). We do not show red arrows (corresponding to e_N action) used in Fig. 6 but they can be easily recovered using the decomposition over TL_N.

linearly independent extensions.

For any sector with non-zero $j = S^z$, we propose the subquotient structure for $\widehat{\mathcal{P}}_j$ as well given in Fig. 7, on the right. The tower for $\widehat{\mathcal{P}}_0$ has the trivial top subquotient because $(d_0^0) = 0$ and cut at the L-th level, i.e. ended by the pair of (d_L^0), while other towers have the top and also ended by the pair of (d_L^0). We note that the two simple subquotients at each level of the ladders are isomorphic. The hamiltonian H from (2.6) acts by Jordan blocks of rank 2 on each pair of isomorphic simple subquotients with one at the top (having only outgoing arrows) and the second subquotient in the socle of the module (having only ingoing arrows). The Jordan block structure is due to presence of zero fermionic modes in the hamiltonian as it is observed in [1].

It is important to note that modules $\widehat{\mathcal{P}}_j$ and $\widehat{\mathcal{P}}_{-j}$ are not isomorphic, otherwise we would have an intertwining operator not from $\mathfrak{Z}_{JTL}$, – the only isomorphic ones are the trivials $\widehat{\mathcal{P}}_{-L} \cong \widehat{\mathcal{P}}_L = (1)$ connected by the action of f^L and e^L from $\mathfrak{Z}_{JTL}$. For each $-L + 1 < j < L - 1$, the module $\widehat{\mathcal{P}}_j$ is fully described by a sequence of parameters $\{x_i\}$ on a complex projective line, $x_i \in \mathbb{CP}^1$, which were introduced after (5.2) and mentioned also in 5.1.2 for a particular case. We leave this characterization for a future work. We only note the reason for such parameters to appear is non-faithfulness of the representation on the spin-chain we consider and (at least) two-dimensionality of the first extension groups in Sec. 5.1. The question about these parameters is actually related to "how to obtain the spin-chain modules by particular quotients of projective JTL_N-modules?". The indecomposable modules we encounter with have no a single cyclic vector (in contrast to the standard modules) and are particular quotients of a direct sum of projective covers over JTL_N. To cover a module $\widehat{\mathcal{P}}_j$, one should take the direct sum $\oplus_{k=j}^{L-1}\mathrm{Proj}_k$ of projective covers Proj_k for each simple JTL_N-module $\mathcal{L}_{k,(-1)^{k+1}}$.

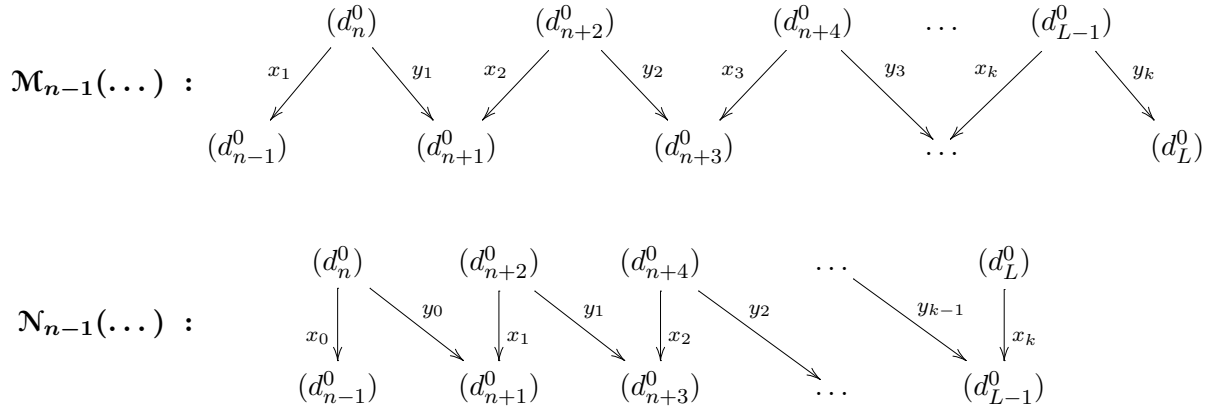$$\mathcal{M}_{n-1}(\dots) \ :$$

$$\mathcal{N}_{n-1}(\dots) \ :$$

Figure 8: For even L, the indecomposable "zig-zag" JTL_N-modules $\mathcal{M}_{n-1}(x_1, y_1, \dots, x_k, y_k)$ at the top, with odd n, and $\mathcal{N}_{n-1}(x_0, y_0, \dots, x_k)$ at the bottom, with even n, $k = (L-\bar{n})/2$, and $\bar{n} = n - (n \bmod 2)$. Each single arrow is a doubled arrow introduced in (5.2) and is characterized by a point on a complex projective line, $x_i, y_i \in \mathbb{CP}^1$.

5.4 The indecomposable "zig-zag" modules

Before describing all homomorphisms between $\widehat{\mathcal{P}}_j$ and $\widehat{\mathcal{P}}_k$, with $-L \leq j, k \leq L$, we introduce indecomposable modules of "zig-zag" shapes. For simplicity, we consider only the case of even L (or $N = 0 \bmod 4$); the odd L case is quite similar. For a positive odd n, let $k = (L-n+1)/2$. Then, taking a quotient of the direct sum $\mathcal{M}_n^{(1)}(x_1, y_1) \oplus \mathcal{M}_{n+2}^{(1)}(x_2, y_2) \oplus \cdots \oplus \mathcal{M}_{L-1}^{(1)}(x_k, y_k)$ of the JTL_N-modules introduced in (5.2) by a submodule $\mathcal{L}_{n+1,(-1)^n} \oplus \mathcal{L}_{n+3,(-1)^n} \oplus \cdots \oplus \mathcal{L}_{L-2,(-1)^n}$ we get a family of JTL_N-modules $\mathcal{M}_{n-1}(x_1, y_1, \dots, x_k, y_k)$, where $x_i, y_i \in \mathbb{CP}^1$. This family is thus parametrized by the set $\{x_i, y_i \,|\, 1 \leq i \leq k\} \in \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$ and presented in Fig. 8, the top one, and we set $(d_n^0) \equiv \mathcal{L}_{n,(-1)^{n+1}}$ as usual and $(d_{L+1}^0) \equiv 0$. We define similarly a family of JTL_N-modules $\mathcal{N}_{n-1}(x_0, y_0, \dots, x_k)$ given at the bottom of Fig. 8, with even n and $k = (L - n)/2$. In what follows, we also use the modules $\mathcal{M}_{n-1}^*(x_1, y_1, \dots, x_k, y_k)$ and $\mathcal{N}_{n-1}^*(x_0, y_0, \dots, x_k)$ with all arrows reversed.

The spin-chain module $\widehat{\mathcal{P}}_0$ is an extension/glueing $\mathcal{N}_1^*(\dots) \to \mathcal{N}_1(\dots)$ of two modules of the $\mathcal{N}$-type, where appropriate parameters stand in the round brackets. As we said before, to determine the parameters is out of the scope of the paper and we will thus use the notation without specifying them explicitly. We only note that a canonical way to specify the submodule $\mathcal{N}_1(\dots)$ in $\widehat{\mathcal{P}}_0$ is to take the kernel of the quantum-group generator F in $\widehat{\mathcal{P}}_0$; we remind that (the representation (2.8) of) F belongs to the JTL_N-centralizer $\mathfrak{Z}_{JTL}$ on the spin-chain and therefore its kernel is a JTL_N-module. That the kernel of F is isomorphic to a $\mathcal{N}_1$ module (with appropriate parameters) easily follows from the decompositions over $U_{\mathsf{q}}s\ell(2)$ in (3.1) and over TL_N in (5.3) restricted to $S^z = 0$ and the explicit action of $U_{\mathsf{q}}s\ell(2)$ given in App. B.

The JTL_N-modules $\widehat{\mathcal{P}}_j$, for $j \neq 0$, proposed on the right side of Fig. 7 can be also obtained as an extension of two modules $\mathcal{N}_{|j|}(\dots)$ and $\mathcal{M}_{|j|+1}(\dots)$, for odd j, with the first one being a subquotient and the second is a submodule of $\widehat{\mathcal{P}}_j$, and similarly for the even-j case. For any j, using again the decompositions (3.1) and (5.3) restricted to the supbspace with $S^z = \pm j$ and the $U_{\mathsf{q}}s\ell(2)$-action from

App. B, we obtain the following short exact sequences of JTL_N-modules

$$0 \rightarrow \mathcal{N}_1(\ldots) \rightarrow \widehat{\mathcal{P}}_0 \rightarrow \mathcal{N}_1^*(\ldots) \rightarrow 0, \tag{5.7}$$

$$0 \rightarrow \mathcal{M}_{|j|+1}(\ldots) \rightarrow \widehat{\mathcal{P}}_j \rightarrow \mathcal{N}_{|j|}(\ldots) \rightarrow 0, \qquad j - \text{odd}, \tag{5.8}$$

$$0 \rightarrow \mathcal{N}_{|j|+1}(\ldots) \rightarrow \widehat{\mathcal{P}}_j \rightarrow \mathcal{M}_{|j|}(\ldots) \rightarrow 0, \qquad j - \text{even}, \tag{5.9}$$

where we define the submodules $\mathcal{M}_{|j|+1}(\ldots)$ and $\mathcal{N}_{|j|+1}(\ldots)$ as the kernels of the quantum-group generator F on $\widehat{\mathcal{P}}_j$, for $j > 0$, and the kernels of E, for $j < 0$, for odd and even j, respectively. The projective-line parameters in the round brackets of all modules in (5.7)-(5.9) are thus uniquely fixed.

Using the self-conjugacy $(e_j^\dagger = e_j$, with $1 \leq j \leq N)$ of the JTL_N-representation $\pi_{g\ell}$ in (2.4) which implies that $\widehat{\mathcal{P}}_j^* \cong \widehat{\mathcal{P}}_j$, we state the dual short exact sequences of JTL_N-modules

$$0 \rightarrow \mathcal{N}_{|j|}^*(\ldots) \rightarrow \widehat{\mathcal{P}}_j \rightarrow \mathcal{M}_{|j|+1}^*(\ldots) \rightarrow 0, \qquad j - \text{odd}, \tag{5.10}$$

$$0 \rightarrow \mathcal{M}_{|j|}^*(\ldots) \rightarrow \widehat{\mathcal{P}}_j \rightarrow \mathcal{N}_{|j|+1}^*(\ldots) \rightarrow 0, \qquad j - \text{even}, \tag{5.11}$$

where the submodules $\mathcal{N}_{|j|}^*(\ldots)$ and $\mathcal{M}_{|j|}^*(\ldots)$ are now defined as the kernels of the quantum-group generator E on $\widehat{\mathcal{P}}_j$, for $j > 0$, and the kernels of F, for $j < 0$, for odd and even j, respectively. Then, parameters in the round brackets in (5.10)-(5.11) are also uniquely fixed.

We next use the zig-zag modules described above and the short exact sequences in description of all homomorphisms between the spin-chain JTL_N-modules proposed above.

Theorem 5.5. *For $-L \leq j, k \leq L$, the space of homomorphisms between $\widehat{\mathcal{P}}_j$ and $\widehat{\mathcal{P}}_k$ has the dimension*

$$\dim \operatorname{Hom}(\widehat{\mathcal{P}}_j, \widehat{\mathcal{P}}_k) = \begin{cases} 1, & j - k = 1 \mod 2, \\ \frac{1}{2}(L - \max(|j|, |k|) + j \mod 2) + \delta_{j,k}, & j - k = 0 \mod 2, \end{cases} \tag{5.12}$$

and the one-dimensional space in the case $j - k = 1 \mod 2$ is given by the map $f_{j,k} \in \operatorname{Hom}(\widehat{\mathcal{P}}_j, \widehat{\mathcal{P}}_k)$ with its image

$$\operatorname{im}(f_{j,k}) \cong \begin{cases} \mathcal{N}_{|j|}(\ldots), & j - odd \ and \ |j| > |k|, \\ \mathcal{M}_{|k|}^*(\ldots), & j - odd \ and \ |j| < |k|, \\ \mathcal{M}_{|j|}(\ldots), & j - even \ and \ |j| > |k|, \\ \mathcal{N}_{|k|}^*(\ldots), & j - even \ and \ |j| < |k|, \end{cases} \tag{5.13}$$

with appropriate parameters from $\mathbb{CP}^1$ standing at the dots in round brackets.

In the case $j - k = 0 \mod 2$, the Hom*-space is spanned by homomorphisms with semisimple images.*

We give only an idea of the proof. The case when $j - k$ is even is obvious and follows from the subquotient structure of $\widehat{\mathcal{P}}_j$ given in Fig. 7: a basis in the space $\operatorname{Hom}(\widehat{\mathcal{P}}_j, \widehat{\mathcal{P}}_k)$ can be chosen as the homomorphisms having the images isomorphic to $\mathcal{L}_{j,(-1)^{j+1}}$.

The case when $j - k$ is odd can be analyzed taking concatenation of the short exact sequences (5.7)-(5.11) with the throughout mappings F and E. The result of such concatenation are two cochain complexes with the differentials F and E (we remind that $\mathsf{F}^2 = \mathsf{E}^2 = 0$)

$$0 \rightarrow \widehat{\mathcal{P}}_L \xrightarrow{\mathsf{F}} \widehat{\mathcal{P}}_{L-1} \xrightarrow{\mathsf{F}} \ldots \xrightarrow{\mathsf{F}} \widehat{\mathcal{P}}_{j+1} \xrightarrow{\mathsf{F}} \widehat{\mathcal{P}}_j \xrightarrow{\mathsf{F}} \widehat{\mathcal{P}}_{j-1} \xrightarrow{\mathsf{F}} \ldots \xrightarrow{\mathsf{F}} \widehat{\mathcal{P}}_{-L} \rightarrow 0, \tag{5.14}$$

$$0 \rightarrow \widehat{\mathcal{P}}_{-L} \xrightarrow{\mathsf{E}} \widehat{\mathcal{P}}_{-L+1} \xrightarrow{\mathsf{E}} \ldots \xrightarrow{\mathsf{E}} \widehat{\mathcal{P}}_{j-1} \xrightarrow{\mathsf{E}} \widehat{\mathcal{P}}_j \xrightarrow{\mathsf{E}} \widehat{\mathcal{P}}_{j+1} \xrightarrow{\mathsf{E}} \ldots \xrightarrow{\mathsf{E}} \widehat{\mathcal{P}}_L \rightarrow 0, \tag{5.15}$$

which have trivial cohomologies, i.e. they are long exact sequences. The images (and therefore the kernels) of these differentials are the zig-zag JTL_N-modules described in (5.7)-(5.11). This proves existence of the homomorphisms $f_{j,j\pm1}$ with the properties (5.13) in the case $j - k = \pm1$. Existence for all other cases is proved taking into account the commutation of JTL_N with operators F_n and E_m from the representation $\rho_{g\ell}$ of $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$. To compute their images, we use the homomorphism of algebras from Rem. 2.4.2 together with the $U_{\mathsf{q}}s\ell(2)$-action given in App. B. Then (5.13) follows from the decompositions over $U_{\mathsf{q}}s\ell(2)$ in (3.1) and over TL_N in (5.3) restricted to the subspaces with $\rho_{g\ell}(\mathsf{h}) = j/2$ and $\rho_{g\ell}(\mathsf{h}) = k/2$.

To prove that there are no other homomorphisms (up to an overall rescaling) between $\widehat{\mathcal{P}}_j$ and $\widehat{\mathcal{P}}_k$, with $j - k$ is an odd number, is sufficient to consider filtrations of the zig-zag submodules/subquotients in $\widehat{\mathcal{P}}_j$ and $\widehat{\mathcal{P}}_k$ by their (smaller) zig-zag submodules, see Fig. 8 where the smaller zig-zag submodules are easily identified, and to use the subquotient structure for $\widehat{\mathcal{P}}_j$ proposed in Fig. 7. Any homomorphism should obviously respect the filtrations. The only care should be taken for a pair of arrows connecting isomorphic pair of subquotients – these arrows correspond to linearly independent elements from the first extension groups in Lem. 5.1.1.

5.6 Intertwiners and the PBW basis

We now identify all the homomorphisms from the space $\mathrm{End}_{JTL_N}(\mathcal{H}_N) = \bigoplus_{j,k=-L}^{L} \mathrm{Hom}(\widehat{\mathcal{P}}_j, \widehat{\mathcal{P}}_k)$ with the PBW basis elements in $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$ that are represented faithfully on the spin-chain. For $j - k$ is an odd number and for each $f_{j,k} \in \mathrm{Hom}(\widehat{\mathcal{P}}_j, \widehat{\mathcal{P}}_k)$ described in Thm. 5.5, we have the equality

$$f_{j,k} = \rho_{g\ell}\big(\mathsf{F}_{(j-k-1)/2}\, p_j(\mathsf{h})\big), \qquad \text{for } j > k, \tag{5.16}$$

$$f_{j,k} = \rho_{g\ell}\big(\mathsf{E}_{(k-j-1)/2}\, p_j(\mathsf{h})\big), \qquad \text{for } j < k, \tag{5.17}$$

where projectors p_j onto $\widehat{\mathcal{P}}_j$ are polynomials in h introduced in (5.6). The homomorphisms $f_{j,k}$ in the case $j - k$ is an even number are identified with $\rho_{g\ell}(p_j(\mathsf{h}))$ if $j = k$ and otherwise with composites of the generators F_n and E_m, times the projector $p_j(\mathsf{h})$. We obtain by a simple calculation that these operators exhaust the PBW basis of $U_{\mathsf{q}}^{\mathrm{odd}}s\ell(2)$ represented faithfully on the spin-chain at even N.

We thus have shown that a sufficient condition on the module structure in Fig. 7 to have the centralizer $\mathfrak{Z}_{JTL}$ indeed holds. To show that the subquotient structure indeed corresponds to the JTL_N action, we do a further and final analysis.

5.7 Final analysis

To finish our proof of correctness of the proposed subquotient structure for $\widehat{\mathcal{P}}_j$, we describe next a subquotient structure for $\widehat{\mathcal{P}}_j$ considered as a module over the centralizer of $\mathfrak{Z}_{JTL}$ which is isomorphic by the definition to the algebra $\mathrm{End}_{\mathfrak{Z}_{JTL}}(\mathcal{H}_N)$. The centralizer obviously contains $\pi_{g\ell}(JTL_N)$ as a subalgebra. The opposite inclusion is not true as we show now.

The subquotient structure can be obtained using intertwining operators respecting $\mathfrak{Z}_{JTL}$ action. These are described in Thm. 3.4.4. The only difference from the diagrams for JTL_N in Fig. 7 is that there are additional ('long') arrows mapping a top subquotient (d_j^0) (having only outgoing arrows) to (d_k^0) in the socle (having only ingoing arrows) whenver $|j - k| \geq 4$ is an even number. We note these long arrows are not composites of any short arrows mapping from the top to the middle level, and from the middle to the socle; this distinguishing property appears only at $N \geq 10$. It turns out that

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JTL_N generators correspond only to these short arrows and not to the long ones, and therefore there is no an element from JTL_N represented by a long arrow. This can be shown using a direct calculation with fermionic expressions for e_j and u^2 (see (3.9) and (3.12) from [1]) in a basis of root vectors of the hamiltonian H from (2.6). Indeed, the expression for e_j is a bilinear combination of $2(N-1)$ generators of a Clifford algebra. A half of them ($N-2$ creation modes $\chi_{p>0}^\dagger$ and $\eta_{p>0}$ in notations of [1], Sec. 4) generates the bottom level – the inrtersection of the kernels of F and E in $\mathcal{H}_N$ – from the vacuum state $\boldsymbol{\Omega}$, and also the top level from one cyclic vector $\boldsymbol{\omega}$ which is involved with $\boldsymbol{\Omega}$ into a Jordan cell for H. Among the Clifford algebra generators, there are two – zero modes η_0 and $\chi_0^\dagger$ – proportional to F and $\mathsf{E}\mathsf{K}^{-1}$, respectively. These are the only generators mapping vectors from the top level to the middle level, and from the middle to the bottom level. We see from the expression (4.22) in [1] that the u^2, which is also a sum of monomials in the Clifford algebra, has no a monomial containing the product of the two zero modes. The product maps the top to the bottom and a monomial containing it could thus correspond to a long arrow. The e_j's have such a monomial but it is quadratic, i.e., proportional to the product of the zero modes, and thus commutes with the JTL_N action and maps a top subquotient (d_j^0) only to the bottom (d_j^0). The fermionic expression for e_j has also other terms/monoms containing only one of the zero modes and they thus map only by one level down. We conclude that the action of e_j, with $1 \le j \le N$, and u^2 can not correspond to that long arrows connecting the top and the bottom and which are not composites of any short arrows. This proves that there are no such arrows in diagrams for the subquotient structure of JTL_N-modules $\widehat{\mathcal{P}}_j$. We can thus conclude that the algebra $\pi_{g\ell}(JTL_N)$ does not contain the double centralizer $\mathrm{End}_{\mathfrak{Z}_{JTL}}(\mathcal{H}_N)$.

Finally, we comment that removing at least one red arrow from the diagrams for $\widehat{\mathcal{P}}_0$ in Fig. 6 or for $\widehat{\mathcal{P}}_j$ in Fig. 7 results in an enlarged endomorphism algebra (black arrows should be present due to the action of the subalgebra TL_N.) Indeed, removing a red arrow mapping from a top subquotient (d_j^0) to (d_{j+1}^0) in the middle we should remove also the red arrow mapping from the same subquotient (d_{j+1}^0) to (d_j^0) in the bottom because of the self-conjugacy $(e_j^\dagger = e_j)$ of the JTL_N-representation $\pi_{g\ell}$ in (2.4) which implies that $\widehat{\mathcal{P}}_j^* \cong \widehat{\mathcal{P}}_j$. Then, we can repeat the same analysis as in Sec. 5.2.1 for $N = 8$ and get an additional intertwining operator not from $\mathfrak{Z}_{JTL}$ but this contradicts to Thm. 2.4.3. Removing a red arrow connecting (d_j^0) and (d_{j-1}^0) results eventually in a contradiction to a statement related to Lem. 5.1.1 in a way very similar to what was stated also in the example for $N = 8$ in Sec. 5.2.1. We do not give a proper generalization of the results for $N = 8$ because of their simplicity and a lack of extra pages. This analysis finishes our proof of the subquotient structure for $\widehat{\mathcal{P}}_j$ modules over JTL_N proposed in Fig. 7.

5.8   Comparison with the standard modules

We finally give a qualitative characterization of the spin-chain modules $\widehat{\mathcal{P}}_j$ in the context of the standard modules in Fig. 4 discussed in Sec. 4.2. The subquotient structure of the JTL_N-modules in the spin chain is obtained by flipping half the arrows in the standard modules of T_N^a and ignoring the subscript $\pm$ (distinguishing only non-isomorphic simple T_N^a-subquotients but not the ones over JTL_N), as illustrated on Fig. 9. This is similar to what happens when comparing Verma and Feigin–Fuchs modules over a Virasoro algebra. Note that we do not use here to the standard modules for JTL_N which turn out to have no arrows inside the tower on Fig. 4; we believe this latter feature is a peculiarity of the case $\mathsf{q} = i$.
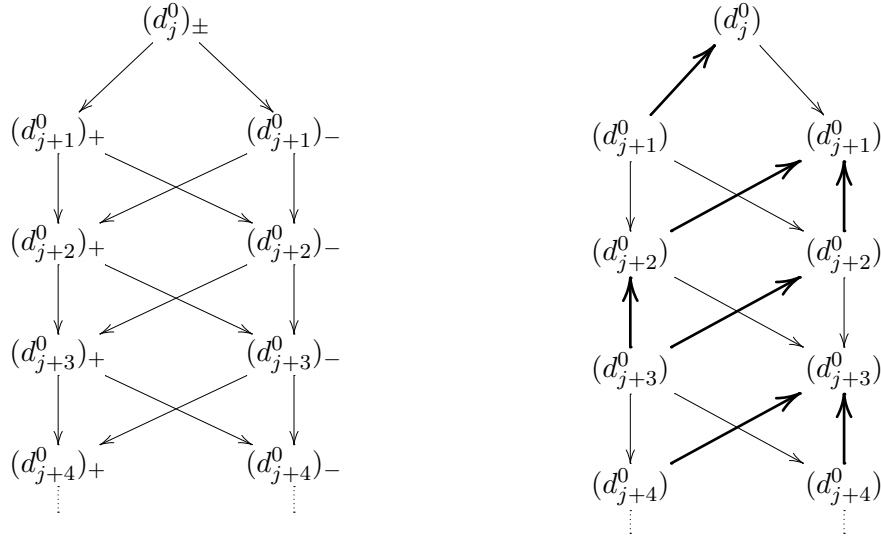
Figure 9: The structure of the spin chain modules $\widehat{\mathcal{P}}_j$ at $\mathfrak{q} = i$ (the right one). The thick arrows have been flipped with respect to the structure of the standard modules on the left side.

6 Bimodule structure in the closed $g\ell(1|1)$ spin-chains

In this short section, we find the subquotient structure of the bimodules $\mathcal{H}_{2L}$ over the pair of the two commuting algebras centralizing each other both in the periodic and antiperiodic $g\ell(1|1)$ spin-chains.

6.1 Bimodule over JTL_N and $\mathfrak{Z}_{JTL}$

We use the spin-chain decomposition (5.4) over JTL_N described in Secs. 5.2 and 5.3 and the intertwining operators from Thm. 5.5 for studying the structure of the bimodule $\mathcal{H}_N$ over the two algebras JTL_N and $\mathfrak{Z}_{JTL}$.

One way to describe the bimodule $\mathcal{H}_{2L}$ is to consider the cochain complexes (5.14) and (5.15) with the differentials F and E. The images (and the kernels) of these differentials and of the generators F_n and E_m are the zig-zag JTL_N-modules described in Sec. 5.4 and Thm. 5.5, with the use of the identifications (5.16) and (5.17). The centralizer $\mathfrak{Z}_{JTL}$ then acts on each of these complexes in a 'long-range' way mapping terms with $S^z = j$ to ones with $S^z = k$ and with the only condition that $|j - k| = 1 \bmod 2$.

We finally give a diagram describing the subquotient structure of the bimodule $\mathcal{H}_{2L}$ over the pair $(JTL_N, \mathfrak{Z}_{JTL})$. The two commuting actions are presented in Fig. 10 where we show a direct sum of the spin-chain modules $\widehat{\mathcal{P}}_j$ over JTL_N. The direct sum is depicted as a (horizontal) sequence of diagrams for $\widehat{\mathcal{P}}_j$ from $j = -L$ on the left to $j = L$ on the right. Each node in the diagram is a simple subquotient over the product $JTL_N \boxtimes U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$. The action of JTL_N is depicted by vertical arrows while the action of $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$ is shown by dotted horizontal lines connecting different JTL_N-modules. We note that the JTL_N-modules $\widehat{\mathcal{P}}_j$ in Fig. 10 are drawn in opposite direction 'from bottom to top' comparing to diagrams in Fig. 7.

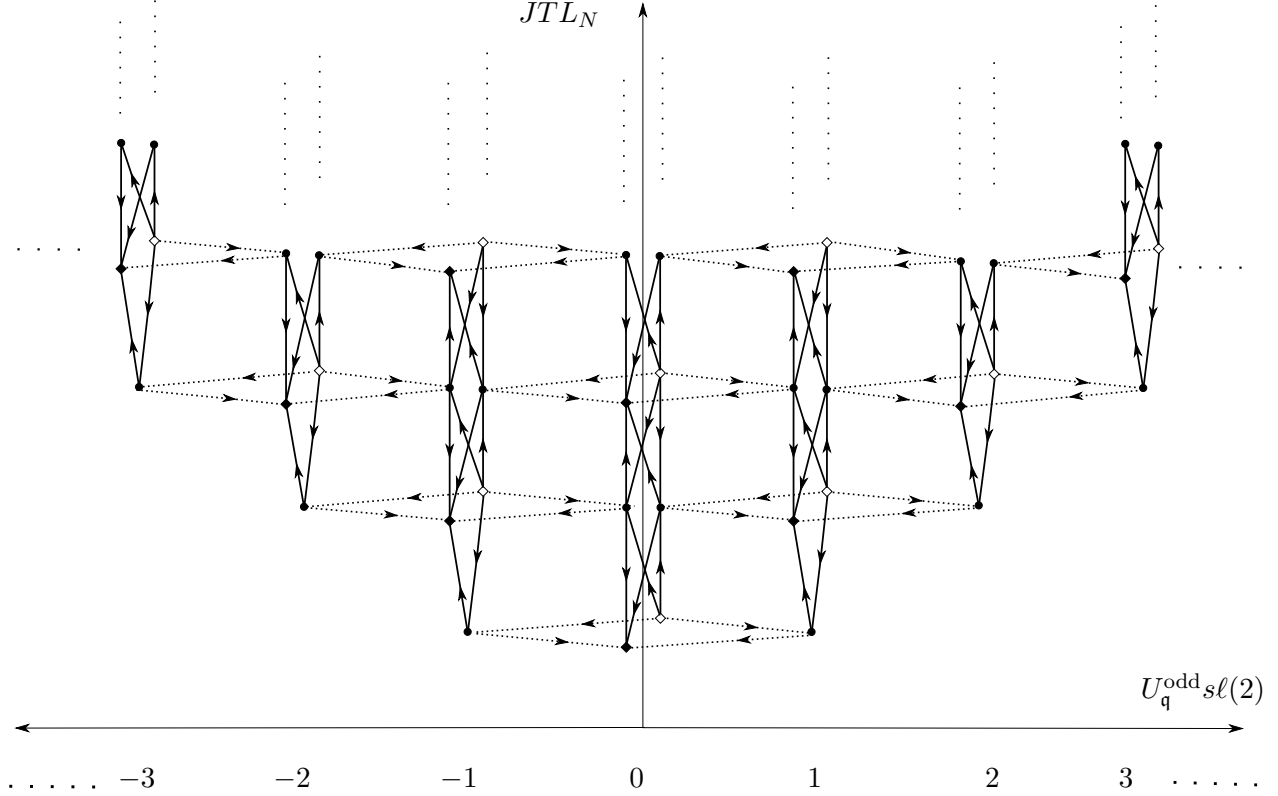In the diagram, the first (horizontal) layer at the bottom contains four nodes, which are simple

Figure 10: Bimodule over the pair $(JTL_N, U_{\mathsf{q}}^{\mathrm{odd}} s\ell(2))$ of commuting algebras. The action of JTL_N is depicted by vertical arrows while the action of $U_{\mathsf{q}}^{\mathrm{odd}} s\ell(2)$ is shown by dotted horizontal lines. Each label j in the horizontal axis corresponds to the sector for S^z and the label runs from $-L$ on the left to $j = L$ on the right. Each vertical tower above a label j is the diagram for $\widehat{\mathcal{P}}_j$. The first horizontal layer at the bottom contains four nodes (d_1^0) and dotted arrows mixing them compose the $U_{\mathsf{q}}^{\mathrm{odd}} s\ell(2)$-module T_1. The second layer contains eight nodes (d_2^0) and the dotted arrows depict the action in the indecomposable module T_2 presented on Fig. 2 in the front of (d_2^0), etc. We suppress long-range arrows representing action of the generators $\mathsf{F}_{>0}$ and $\mathsf{E}_{>0}$ in order to simplify diagrams. For example, the second layer of the bimodule contains in addition four long arrows going from the node $\diamond$ at $j = \mp 1$ to the node $\bullet$ at $j = \pm 2$, and from the node $\bullet$ at $j = \pm 2$ to the node $\blacklozenge$ at $j = \mp 1$.

JTL_N-modules (d_1^0), and dotted arrows mixing them describe the indecomposable $U_q^{\text{odd}} s\ell(2)$-module T_1. The second layer contains eight nodes of type (d_2^0) and the dotted arrows contribute to the indecomposable module T_2 presented on Fig. 2 in the front of (d_2^0), etc. We emphasize that we do not draw long-range arrows representing action of the generators $\mathsf{F}_{>0}$ and $\mathsf{E}_{>0}$ in modules $\mathsf{T}_{n>1}$ in order to simplify diagrams but the arrows can be easily recovered using either the homomorphisms of JTL_N-modules described above in Thm. 5.5 or from the subquotient structure of T_n described in 3.3 – for example, the second layer of the bimodule contains in addition four long arrows going from the node $\diamond$ at $j = \mp 1$ to the node $\bullet$ at $j = \pm 2$, and from the node $\bullet$ at $j = \pm 2$ to the node $\blacklozenge$ at $j = \mp 1$. With this comment about arrows in mind, the reader can compare complexity of this bimodule with the open-case bimodule in Fig. 1.

6.2   The bimodule in the twisted case

We recall [1] that the twisted or antiperiodic model for the $g\ell(1|1)$ spin chain is obtained by setting $f_{2L+1}^{(\dagger)} = -f_1^{(\dagger)}$ (compare with the conditions (2.5) for the periodic model). Then, we obtain from (2.4) a different expression for e_{2L},

$$e_{2L} = (f_{2L} - f_1)(f_{2L}^\dagger - f_1^\dagger),$$

which does not provide more a representation of the JTL_N algebra but does of the even affine Temperley–Lieb algebra O_N introduced in Sec. 2.1, see also (2.3). We recall that in the diagrammatic language the JTL_N algebra corresponds to a quotient of O_N, where a non-contractible loop on a cylinder is replaced by the numerical factor $m = 0$, while the antiperiodic boundary conditions now require a quotient of O_N, where non-contractible loops are given the weight 2 (the dimension of the fundamental or its dual, instead of the superdimension). We also have the relation $u^N = (-1)^j$ which is satisfied in the sector with $2j$ through-lines and which means that we impose the condition $z^{2j} = (-1)^j$ on the z^2-parameter in this sector. We will call the corresponding finite-dimensional algebra JTL_N^{tw}. This algebra reminds us a twisted or deformed version of the Jones algebra studied in [10].

We next recall the result [1] about the centralizer of the representation of JTL_{2L}^{tw}. The choice of an "even" subalgebra in $U_q s\ell(2)$ at generic q, i.e., generated by the renormalized $even$-powers of the E and F gives in the limit $q \to i$ the centralizer for (the representation of) JTL_N^{tw} on the antiperiodic spin-chain — the usual $U(s\ell(2))$ generated by the e and f.

Theorem 6.2.1. *[1] On the alternating antiperiodic $g\ell(1|1)$ spin chain, the centralizer of the image of the representation of the algebra JTL_N^{tw} is the associative algebra $\rho_{g\ell}(U s\ell(2))$.*

We then describe decomposition of the spin-chain over the $U s\ell(2)$ and then use it for a decomposition over JTL_N^{tw}. Recall first the decomposition (3.1) of $\mathcal{H}_N$ over $U_q s\ell(2)$ where each indecomposable direct summand $\mathsf{P}_{1,j}$ given in (3.2) is decomposed over the $U s\ell(2)$ subalgebra onto the direct sum $2\mathsf{X}_{1,j} \oplus \mathsf{X}_{1,j-1} \oplus \mathsf{X}_{1,j+1}$. We remind that each module $\mathsf{X}_{1,j}$ has a trivial action of E, F, and K while it is the j-dimensional $U s\ell(2)$-module. We thus can easily write a decomposition with respect to the action of the renormalized powers e and f:

$$\mathcal{H}_N|_{U s\ell(2)} = \bigoplus_{j=1}^{L+1} \left(2d_j^0 + d_{j-1}^0 + d_{j+1}^0\right)\mathsf{X}_{1,j} \tag{6.1}$$

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where we set $d_0^0 = 0$ and $d_j^0 = 0$ for all $j > L$. The multiplicities in front of $\mathsf{X}_{1,j+1}$, with $0 \leq j \leq L$, give dimensions of simple modules over JTL_N^{tw} which we denote as $\mathcal{L}_{j,(-1)^j}$ (a half of them, those corresponding to even j only, are also modules over JTL_N and we use the same notation which should not confuse). Therefore, we obtain the bimodule structure which is semisimple:

$$\mathcal{H}_N|_{JTL_N^{tw} \boxtimes Us\ell(2)} = \bigoplus_{j=0}^{L} \mathcal{L}_{j,(-1)^j} \boxtimes \mathsf{X}_{1,j+1} \tag{6.2}$$

where the dimension of $\mathcal{L}_{j,(-1)^j}$ is $2d_{j+1}^0 + d_j^0 + d_{j+2}^0$ and is computed using the binomial expression $d_j^0 = \sum_{i=j}^{L} (-1)^{j-i} \left(\binom{N}{L+i} - \binom{N}{L+i+1} \right)$ with the result

$$\dim \mathcal{L}_{j,(-1)^j} = \binom{N}{L+j} - \binom{N}{L+j+2}, \qquad 0 \leq j \leq L. \tag{6.3}$$

This result agrees with the structure of the standard modules $\mathcal{W}_{j,(-1)^j}$ over JTL_N^{tw} that can be deduced using [5]. The subquotient structure is simpler than for $\mathcal{W}_{j,(-1)^{j+1}}$ JTL_N-modules, which are of the two-strands type described in Sec. 4.2, and is now of a chain type:

$$\mathcal{W}_{j,(-1)^j} : \qquad \mathcal{L}_{j,(-1)^j} \longrightarrow \mathcal{L}_{j+2,(-1)^j} \longrightarrow \mathcal{L}_{j+4,(-1)^j} \longrightarrow \dots \tag{6.4}$$

Recal that $\dim \mathcal{W}_{j,z^2} = \binom{N}{L+j}$. Then, the dimensions (6.3) correspond to single subtractions in accordance with the subquotient structure (6.4).

7   Conclusion

At the end of this technical paper we have thus reached our goal of obtaining the bimodule structure for the $g\ell(1|1)$ spin chain. While the results are somewhat more complicated than in the open case, they nevertheless bear a strong similarity with it. This corresponds closely with the fact that bulk and boundary symplectic fermions theories are deeply related as well. We emphasize that this is a feature particular to the $g\ell(1|1)$ case, which provides a non faithful representation of the Jones–Temperley–Lieb algbera. Cases such as $g\ell(2|2)$ would be faithful, and in a certain sense even more complicated, even though faithfulness would make many technical aspects in fact simpler.

Our next and crucial task is to compare the bimodule over JTL_N and $U_q^{odd} s\ell(2)$ with the known information about the bulk symplectic fermion theory, and see to what extent the algebraic properties of the finite spin chain could have been used to infer those of the continuum limit. This will be discussed in the third paper of this series [4].

Acknowledgements

Appendix A: The full quantum group $U_{\mathfrak{q}}s\ell(2)$ at roots of unity

Here, we collect standard expressions for the quantum group $U_{\mathfrak{q}}s\ell(2)$ we use in the analysis of symmetries of $g\ell(1|1)$ spin-chains. This appendix is identical to appendix A in our first paper [1], and reproduced here only for the reader's convenience. We introduce standard notation for $\mathfrak{q}$-numbers $[n] = \frac{\mathfrak{q}^n - \mathfrak{q}^{-n}}{\mathfrak{q} - \mathfrak{q}^{-1}}$ and $[n]! = [1][2]\dots[n]$.

A.1 Defining relations

The *full* (or Lusztig) quantum group $U_{\mathfrak{q}}s\ell(2)$ with $\mathfrak{q} = e^{i\pi/p}$, for integer $p \geq 2$, is generated by E, F, and K satisfying the standard relations for the quantum $s\ell(2)$,

$$\mathsf{K}\mathsf{E}\mathsf{K}^{-1} = \mathfrak{q}^2\mathsf{E}, \quad \mathsf{K}\mathsf{F}\mathsf{K}^{-1} = \mathfrak{q}^{-2}\mathsf{F}, \quad [\mathsf{E}, \mathsf{F}] = \frac{\mathsf{K} - \mathsf{K}^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}}, \tag{A1}$$

with some constraints,

$$\mathsf{E}^p = \mathsf{F}^p = 0, \quad \mathsf{K}^{2p} = \mathbf{1}, \tag{A2}$$

and additionally by the divided powers $\mathsf{f} \sim \mathsf{F}^p/[p]!$ and $\mathsf{e} \sim \mathsf{E}^p/[p]!$, which turn out to satisfy the usual $s\ell(2)$-relations:

$$[\mathsf{h}, \mathsf{e}] = \mathsf{e}, \qquad [\mathsf{h}, \mathsf{f}] = -\mathsf{f}, \qquad [\mathsf{e}, \mathsf{f}] = 2\mathsf{h}. \tag{A3}$$

There are also 'mixed' relations

$$[\mathsf{h}, \mathsf{K}] = 0, \qquad [\mathsf{E}, \mathsf{e}] = 0, \qquad [\mathsf{K}, \mathsf{e}] = 0, \qquad [\mathsf{F}, \mathsf{f}] = 0, \qquad [\mathsf{K}, \mathsf{f}] = 0, \tag{A4}$$

$$[\mathsf{F}, \mathsf{e}] = \frac{1}{[p-1]!}\mathsf{K}^p \frac{\mathfrak{q}\mathsf{K} - \mathfrak{q}^{-1}\mathsf{K}^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}}\mathsf{E}^{p-1}, \qquad [\mathsf{E}, \mathsf{f}] = \frac{(-1)^{p+1}}{[p-1]!}\mathsf{F}^{p-1} \frac{\mathfrak{q}\mathsf{K} - \mathfrak{q}^{-1}\mathsf{K}^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}}, \tag{A5}$$

$$[\mathsf{h}, \mathsf{E}] = \frac{1}{2}\mathsf{E}A, \quad [\mathsf{h}, \mathsf{F}] = -\frac{1}{2}A\mathsf{F}, \tag{A6}$$

where

$$A = \sum_{s=1}^{p-1} \frac{(u_s(\mathfrak{q}^{-s-1}) - u_s(\mathfrak{q}^{s-1}))\mathsf{K} + \mathfrak{q}^{s-1}u_s(\mathfrak{q}^{s-1}) - \mathfrak{q}^{-s-1}u_s(\mathfrak{q}^{-s-1})}{(\mathfrak{q}^{s-1} - \mathfrak{q}^{-s-1})u_s(\mathfrak{q}^{-s-1})u_s(\mathfrak{q}^{s-1})} u_s(\mathsf{K})\boldsymbol{e}_s \tag{A7}$$

with the polynomials $u_s(\mathsf{K}) = \prod_{n=1,\,n\neq s}^{p-1}(\mathsf{K} - \mathfrak{q}^{s-1-2n})$, and $\boldsymbol{e}_s$ are some central primitive idempotents [15]. The relations (A1)-(A7) are the defining relations of the associative algebra $U_{\mathfrak{q}}s\ell(2)$.

The quantum group $U_{\mathfrak{q}}s\ell(2)$ has the Hopf-algebra structure with the comultiplication

$$\Delta(\mathsf{E}) = \mathbf{1} \otimes \mathsf{E} + \mathsf{E} \otimes \mathsf{K}, \quad \Delta(\mathsf{F}) = \mathsf{K}^{-1} \otimes \mathsf{F} + \mathsf{F} \otimes \mathbf{1}, \quad \Delta(\mathsf{K}) = \mathsf{K} \otimes \mathsf{K}, \tag{A8}$$

$$\Delta(\mathsf{e}) = \mathsf{e} \otimes \mathbf{1} + \mathsf{K}^p \otimes \mathsf{e} + \frac{1}{[p-1]!}\sum_{r=1}^{p-1} \frac{\mathfrak{q}^{r(p-r)}}{[r]}\mathsf{K}^p\mathsf{E}^{p-r} \otimes \mathsf{E}^r\mathsf{K}^{-r}, \tag{A9}$$

$$\Delta(\mathsf{f}) = \mathsf{f} \otimes \mathbf{1} + \mathsf{K}^p \otimes \mathsf{f} + \frac{(-1)^p}{[p-1]!}\sum_{s=1}^{p-1} \frac{\mathfrak{q}^{-s(p-s)}}{[s]}\mathsf{K}^{p+s}\mathsf{F}^s \otimes \mathsf{F}^{p-s}. \tag{A10}$$

The antipode and counity are not used in the paper but a reader can find them, for exmaple, in [15].

We can easily write the $(N-1)$-folded coproduct for the capital generators E and F,

$$\Delta^{N-1}\mathsf{E} = \sum_{j=1}^{N} \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{j-1} \otimes \mathsf{E} \otimes \mathsf{K} \otimes \ldots \otimes \mathsf{K}, \qquad \Delta^{N-1}\mathsf{F} = \sum_{j=1}^{N} \underbrace{\mathsf{K}^{-1} \otimes \ldots \otimes \mathsf{K}^{-1}}_{j-1} \otimes \mathsf{F} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}. \quad \text{(A11)}$$

A.2 Standard spin-chain notations

We introduced the more usual (in the spin-chain literature [17, 18][5]) quantum group generators

$$S^{\pm} = \sum_{1 \leq j \leq N} \mathsf{q}^{-\sigma_1^z/2} \otimes \ldots \mathsf{q}^{-\sigma_{j-1}^z/2} \otimes \sigma_j^{\pm} \otimes \mathsf{q}^{\sigma_{j+1}^z/2} \otimes \ldots \otimes \mathsf{q}^{\sigma_N^z/2},$$

$$k = \mathsf{q}^{S^z}, \qquad \text{with} \quad S^z = \frac{1}{2}\sum_{j=1}^{2L} \sigma_j^z,$$

where $\sigma_j^{\pm}$ and σ_j^z are 2×2-matrices acting on a jth tensorand,

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A12}$$

The defining relations are then (for $\mathsf{q} = e^{i\pi/p}$ and integer $p \geq 2$)

$$kS^{\pm}k^{-1} = \mathsf{q}^{\pm 1}S^{\pm}, \qquad \left[S^+, S^-\right] = \frac{k^2 - k^{-2}}{\mathsf{q} - \mathsf{q}^{-1}},$$

$$(S^{\pm})^p = 0, \qquad k^{4p} = \mathbf{1},$$

and the comultiplication is

$$\Delta(S^{\pm}) = k^{-1} \otimes S^{\pm} + S^{\pm} \otimes k, \qquad \Delta(k^{\pm 1}) = k^{\pm 1} \otimes k^{\pm 1}.$$

Then, we have the Hopf-algebra homomorphism

$$\mathsf{E} \mapsto S^+ k, \qquad \mathsf{F} \mapsto k^{-1}S^-, \qquad \mathsf{K} \mapsto k^2$$

relating the two dictionaries. The antipode and counity formulas can be easily obtained in the spin-chain notations as well but we do not need them in the paper.

A.2.1 The case of XX spin-chains

For $p = 2$ or "XX spin-chain" case, the $(N-1)$-folded coproduct of the renormalized powers e and f reads

$$\Delta^{N-1}\mathsf{e} = \sum_{j=1}^{N} \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{j-1} \otimes \mathsf{e} \otimes \mathsf{K}^2 \otimes \ldots \otimes \mathsf{K}^2 +$$

$$+ \mathsf{q} \sum_{t=0}^{N-2} \sum_{j=1}^{N-1-t} \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{j-1} \otimes \mathsf{E} \otimes \underbrace{\mathsf{K} \otimes \ldots \otimes \mathsf{K}}_{t} \otimes \mathsf{EK} \otimes \mathsf{K}^2 \otimes \ldots \otimes \mathsf{K}^2 \quad \text{(A13)}$$

---

[5]We note that our convention for the spin-chain representation differs from the one in [17] by the change $\mathsf{q} \to \mathsf{q}^{-1}$.

and

$$\Delta^{N-1}\mathsf{f} = \sum_{j=1}^{N} \underbrace{\mathsf{K}^2 \otimes \ldots \otimes \mathsf{K}^2}_{j-1} \otimes \mathsf{f} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} +$$

$$+ \mathfrak{q}^{-1} \sum_{t=0}^{N-2} \sum_{j=1}^{N-1-t} \underbrace{\mathsf{K}^2 \otimes \ldots \otimes \mathsf{K}^2}_{j=1} \otimes \mathsf{K}^{-1}\mathsf{F} \otimes \underbrace{\mathsf{K}^{-1} \otimes \ldots \otimes \mathsf{K}^{-1}}_{t} \otimes \mathsf{F} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}. \quad \text{(A14)}$$

These renormalized powers can also be expressed in terms of the more usual spin-chain operators, and one finds at $p=2$

$$\Delta^{N-1}(\mathsf{e}) = \mathfrak{q}S^{+(2)}k^2, \qquad \Delta^{N-1}(\mathsf{f}) = \mathfrak{q}^{-1}k^{-2}S^{-(2)},$$

where $\mathfrak{q} = i$ and

$$S^{\pm(2)} = \sum_{1 \le j < k \le N-1} \mathfrak{q}^{-\sigma_1^z} \otimes \ldots \otimes \mathfrak{q}^{-\sigma_{j-1}^z} \otimes \sigma_j^{\pm} \otimes 1 \otimes \ldots \otimes 1 \otimes \sigma_k^{\pm} \otimes \mathfrak{q}^{\sigma_{k+1}^z} \otimes \ldots \otimes \mathfrak{q}^{\sigma_N^z}. \quad \text{(A15)}$$

We also note that the $g\ell(1|1)$ spin-chain representation $\pi_{g\ell}$ is equivalent [1] to a twisted XX spin chain representation π_{XX} of JTL_{2L}. The expression of the Temperley–Lieb generators in this case is well known for the open chain [17],

$$\pi_{\text{XX}}(e_j) \equiv e_j^{XX} = -\frac{1}{2}\left[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y - \mathfrak{q}(\sigma_j^z - \sigma_{j+1}^z)\right], \quad \text{(A16)}$$

where $\sigma^{x,y,z}$ are Pauli matrices introduced in (A12) where we use $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm i\sigma^y)$. To get an equivalence in the closed case we need to set in the expression of e_{2L} the following condition:

$$\sigma_{2L+1}^{\pm} = -(-1)^{S^z}\sigma_1^{\pm}. \quad \text{(A17)}$$

This means that a periodic $g\ell(1|1)$ (alternating) spin chain corresponds to a periodic XX spin chain for odd values of the spin operator S^z and to an antiperiodic XX spin chain for even values of the spin.

Appendix B: Projective $U_\mathfrak{q}s\ell(2)$-modules $\mathsf{P}_{1,r}$

We recall [15] $U_\mathfrak{q}s\ell(2)$-action (for $\mathfrak{q} = i$) in projective covers $\mathsf{P}_{1,r}$ of simple modules $\mathsf{X}_{1,r}$, where r is an integer and $r \ge 1$. A module $\mathsf{X}_{1,r}$ is r-dimensional and spanned by x_m, $0 \le m \le r-1$, with the action[6]

$$\mathsf{E}\,\mathsf{x}_m = \mathsf{F}\,\mathsf{x}_m = 0, \qquad \mathsf{K}\,\mathsf{x}_m = (-1)^{r-1}\mathsf{x}_m,$$
$$\mathsf{h}\,\mathsf{x}_m = \frac{1}{2}(r-1-2m)\mathsf{x}_m, \quad \mathsf{e}\,\mathsf{x}_m = m(r-m)\mathsf{x}_{m-1}, \quad \mathsf{f}\,\mathsf{x}_m = \mathsf{x}_{m+1}, \quad \text{(B1)}$$

where we set $\mathsf{x}_{-1} = \mathsf{x}_r = 0$. For $r = 0$, we also set $\mathsf{X}_{1,0} \equiv 0$. The subquotient structure of $\mathsf{P}_{1,r}$ is then given as

$$\text{(B2)}$$

$$\mathsf{P}_{1,r} \quad = \quad \begin{array}{ccc} & \mathsf{X}_{1,r} & \\ \swarrow & & \searrow \\ \mathsf{X}_{1,r-1} & & \mathsf{X}_{1,r+1} \\ \searrow & & \swarrow \\ & \mathsf{X}_{1,r} & \end{array}$$

---

[6]We simplify a notation used in [15] assuming $\mathsf{X}_{1,r} \equiv \mathsf{X}_{1,r}^{\alpha(r)}$ with $\alpha(r) = (-1)^{r-1}$, and the same for $\mathsf{P}_{1,r}$.

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For $r > 1$, the projective module $\mathsf{P}_{1,r}$ has the basis

$$\{\mathsf{t}_m, \mathsf{b}_m\}_{0 \leq m \leq r-1} \cup \{\mathsf{l}_l\}_{1 \leq l \leq r-1} \cup \{\mathsf{r}_l\}_{0 \leq l \leq r}, \tag{B3}$$

where $\{\mathsf{t}_m\}_{0 \leq m \leq r-1}$ is the basis corresponding to the top module in (B2), $\{\mathsf{b}_m\}_{0 \leq m \leq r-1}$ to the bottom, $\{\mathsf{l}_l\}_{1 \leq l \leq r-1}$ to the left, and $\{\mathsf{r}_l\}_{0 \leq l \leq r}$ to the right module. For $r = 1$, the basis does not contain $\{\mathsf{l}_l\}_{1 \leq l \leq r-1}$ terms and we imply $\mathsf{l}_l \equiv 0$ in the action.

We set $\alpha(r) = (-1)^{r-1}$. The $U_{\mathsf{q}}s\ell(2)$-action on $\mathsf{P}_{1,r}$ is then given by

$$
\begin{aligned}
&\mathsf{K}\mathsf{t}_m = \alpha(r)\mathsf{t}_m, \qquad \mathsf{K}\mathsf{b}_m = \alpha(r)\mathsf{b}_m, && 0 \leq m \leq r-1, \\
&\mathsf{K}\mathsf{l}_l = -\alpha(r)\mathsf{l}_l, && 1 \leq l \leq r-1, \\
&\mathsf{K}\mathsf{r}_l = -\alpha(r)\mathsf{r}_l, && 0 \leq l \leq r, \\
&\mathsf{E}\mathsf{t}_m = \alpha(r)\frac{r-m}{r}\mathsf{r}_m + \alpha(r)\frac{m}{r}\mathsf{l}_m, \qquad \mathsf{E}\mathsf{b}_m = 0, && 0 \leq m \leq r-1, \\
&\mathsf{E}\mathsf{l}_l = \alpha(r)(l-r)\mathsf{b}_{l-1}, && 1 \leq l \leq r-1, \\
&\mathsf{E}\mathsf{r}_l = \alpha(r)l\mathsf{b}_{l-1}, && 0 \leq l \leq r, \\
&\mathsf{F}\mathsf{t}_m = \frac{1}{r}\mathsf{r}_{m+1} - \frac{1}{r}\mathsf{l}_{m+1}, \qquad \mathsf{F}\mathsf{b}_m = 0 && 0 \leq m \leq r-1, \quad (\mathsf{l}_r \equiv 0), \\
&\mathsf{F}\mathsf{l}_l = \mathsf{b}_l, && 1 \leq l \leq r-1, \\
&\mathsf{F}\mathsf{r}_l = \mathsf{b}_l, && 0 \leq l \leq r.
\end{aligned}
\tag{B4}
$$

In thus introduced basis, the $s\ell(2)$-generators e, f and h act in $\mathsf{P}_{1,r}$ as in the direct sum $\mathsf{X}_{1,r} \oplus \mathsf{X}_{1,r-1} \oplus \mathsf{X}_{1,r+1} \oplus \mathsf{X}_{1,r}$ with the action defined in (B1).

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